

Canonical approach to the finite density QCD with winding number expansion

Yusuke Taniguchi (University of Tsukuba)
for
Zn Collaboration



Zn Collaboration

- Six members to study canonical approach

R.Fukuda (The University of Tokyo)

A.Nakamura (Hiroshima University)

S.Oka (Rikkyo University)

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Introduction

Grand canonical ensemble

$$Z_G(T, \mu, V) = \text{Tr} \left[\exp \left(-\frac{1}{T} (\hat{H} - \mu \hat{N}) \right) \right]$$

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for every energy and number of particles

For QCD $[\hat{H}, \hat{N}] = 0$

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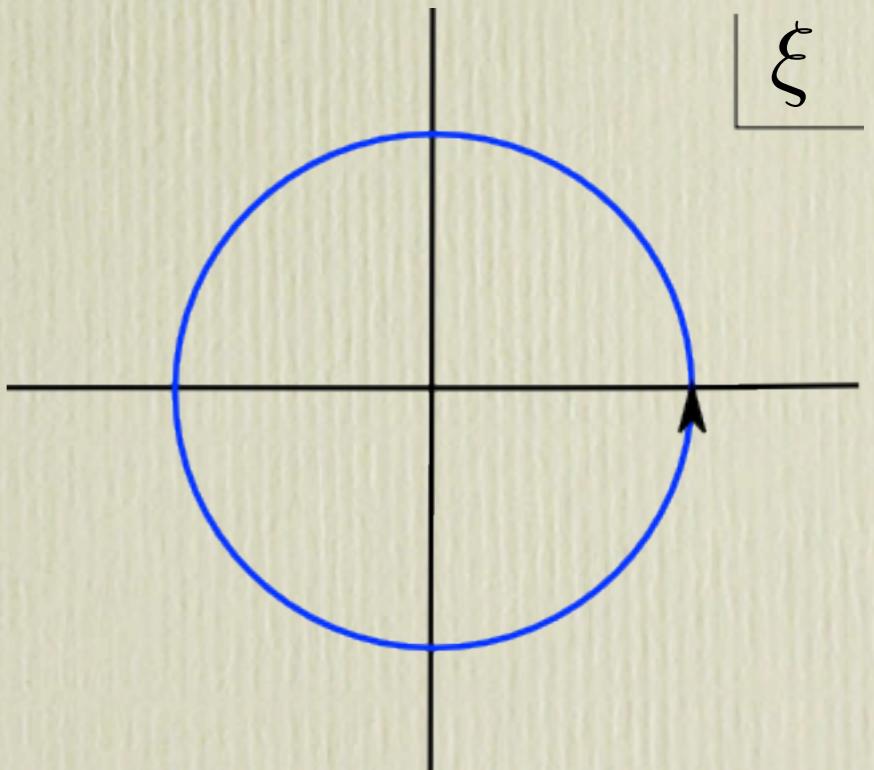
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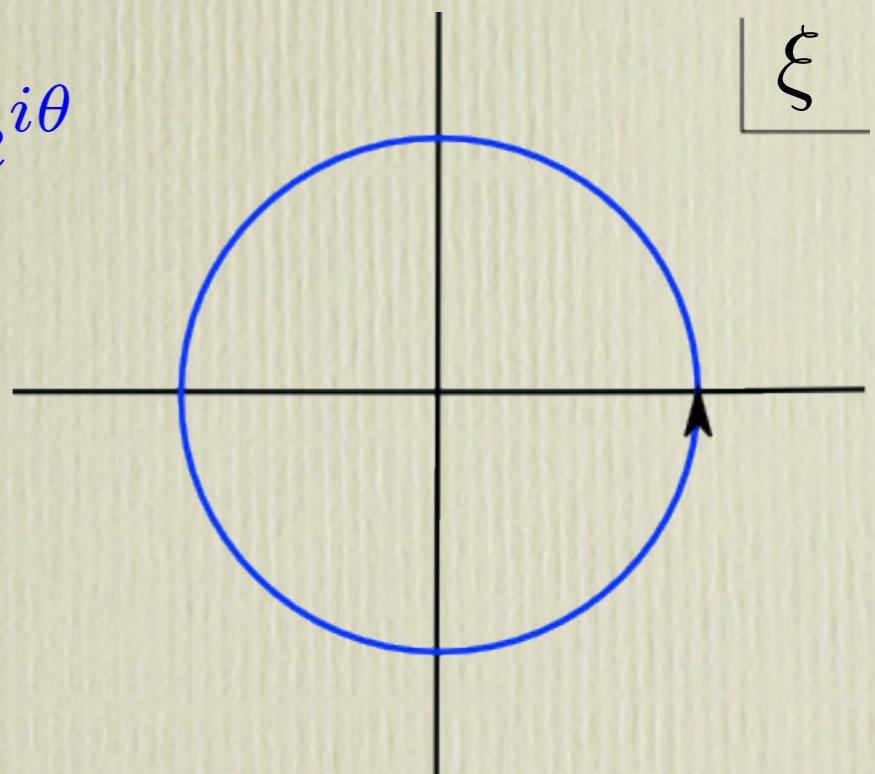
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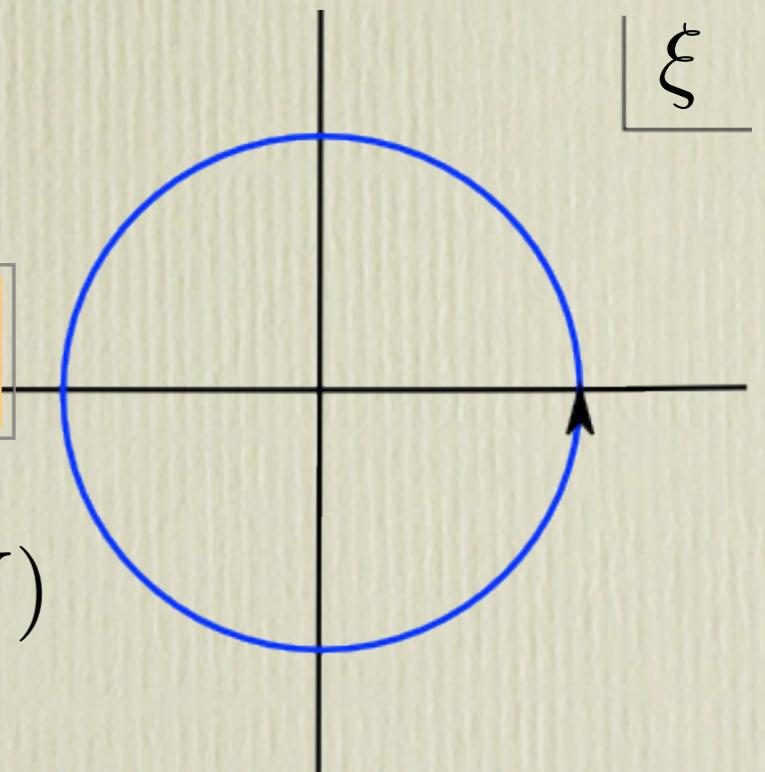
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A. Hasenfratz and D. Toussaint

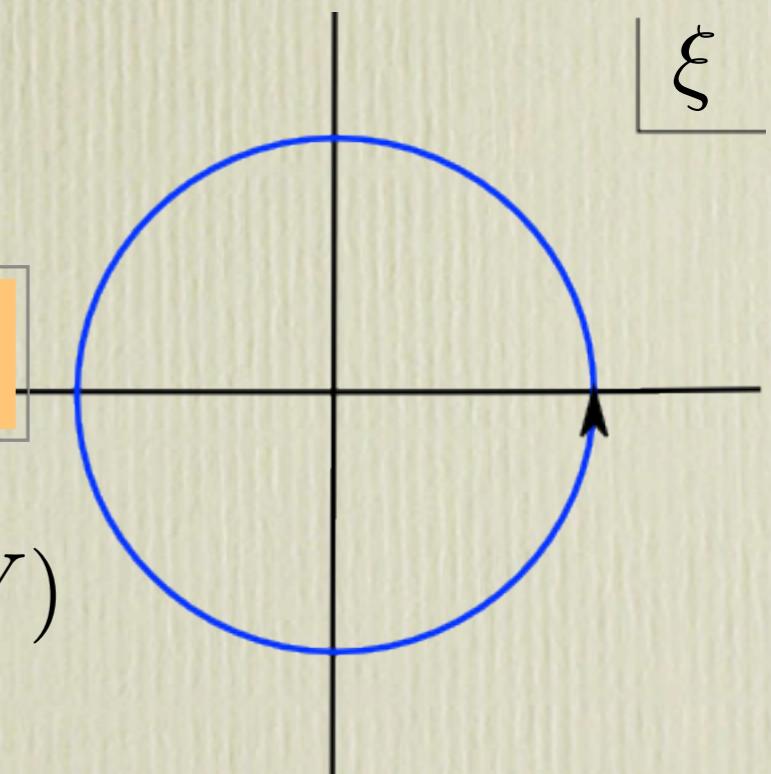
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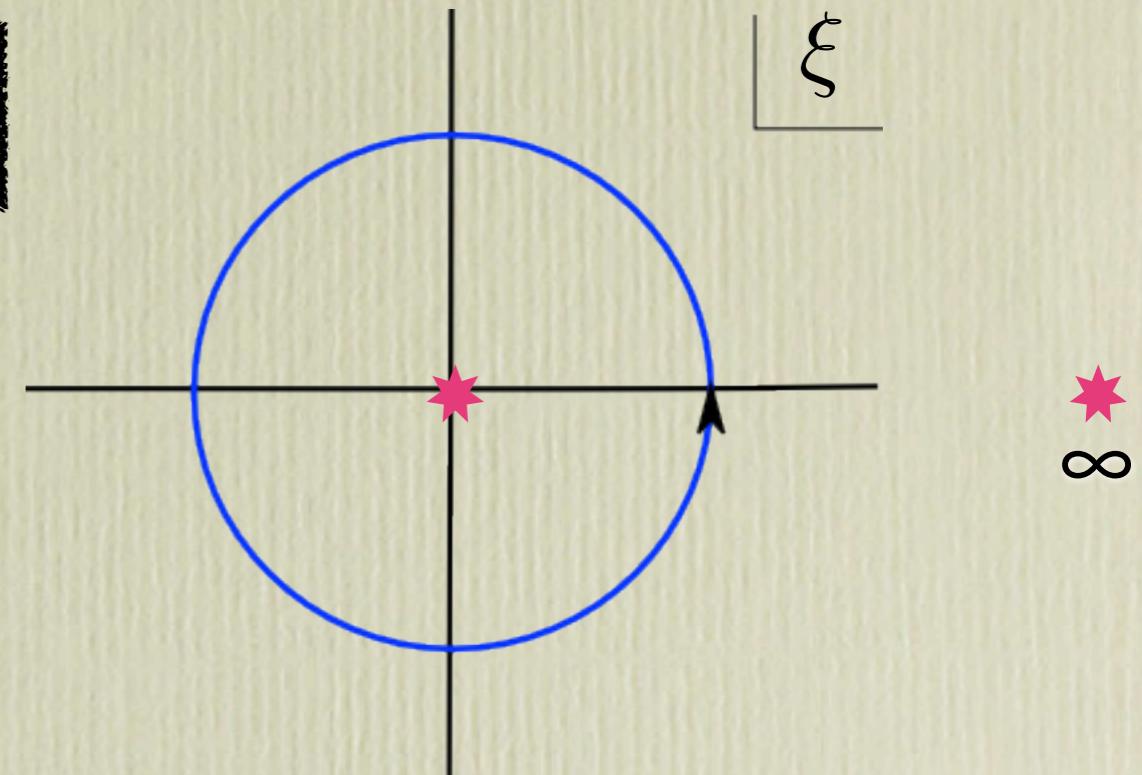
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Singularities only at $\xi=0$ and ∞

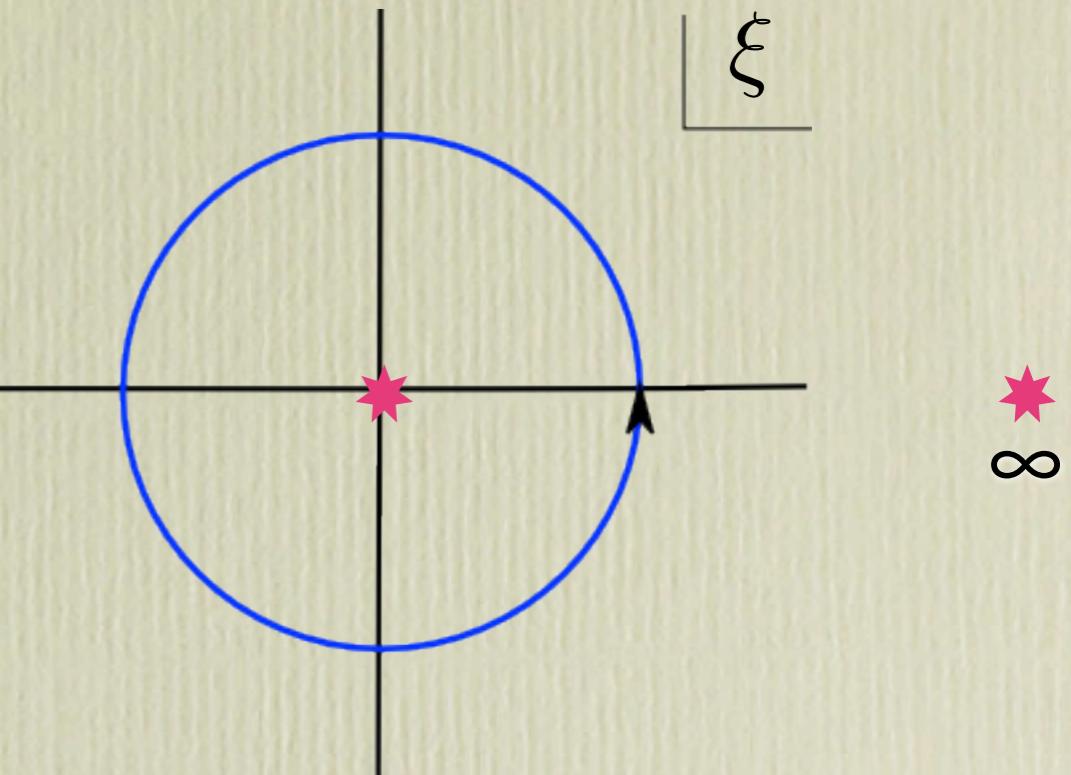


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Making use of the singularity at $\xi=0$



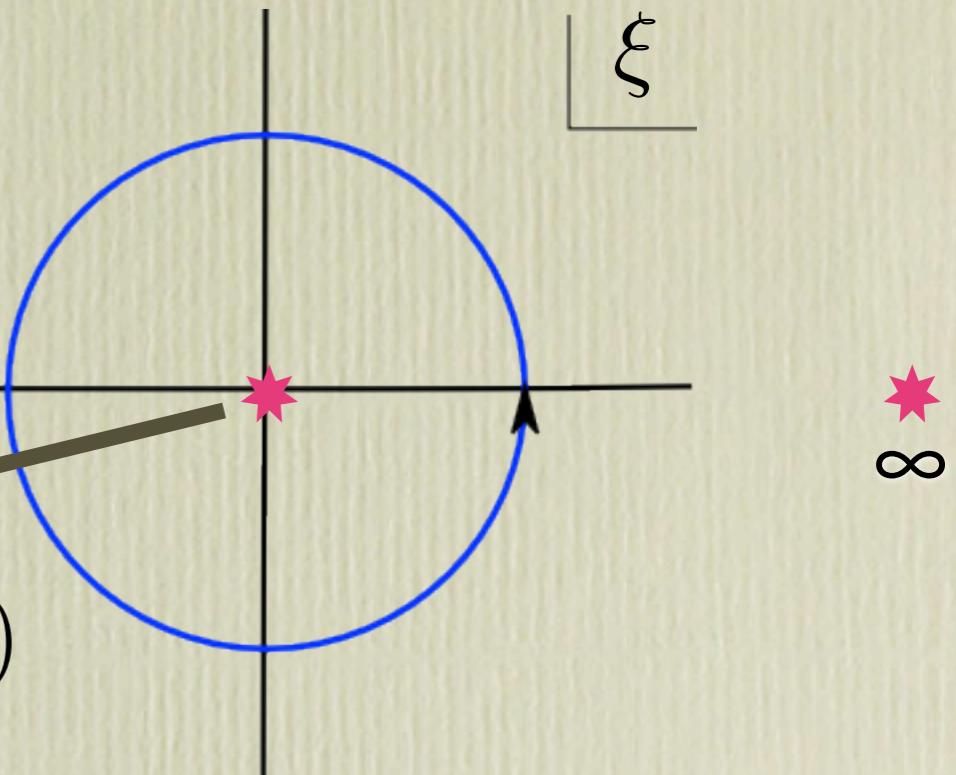
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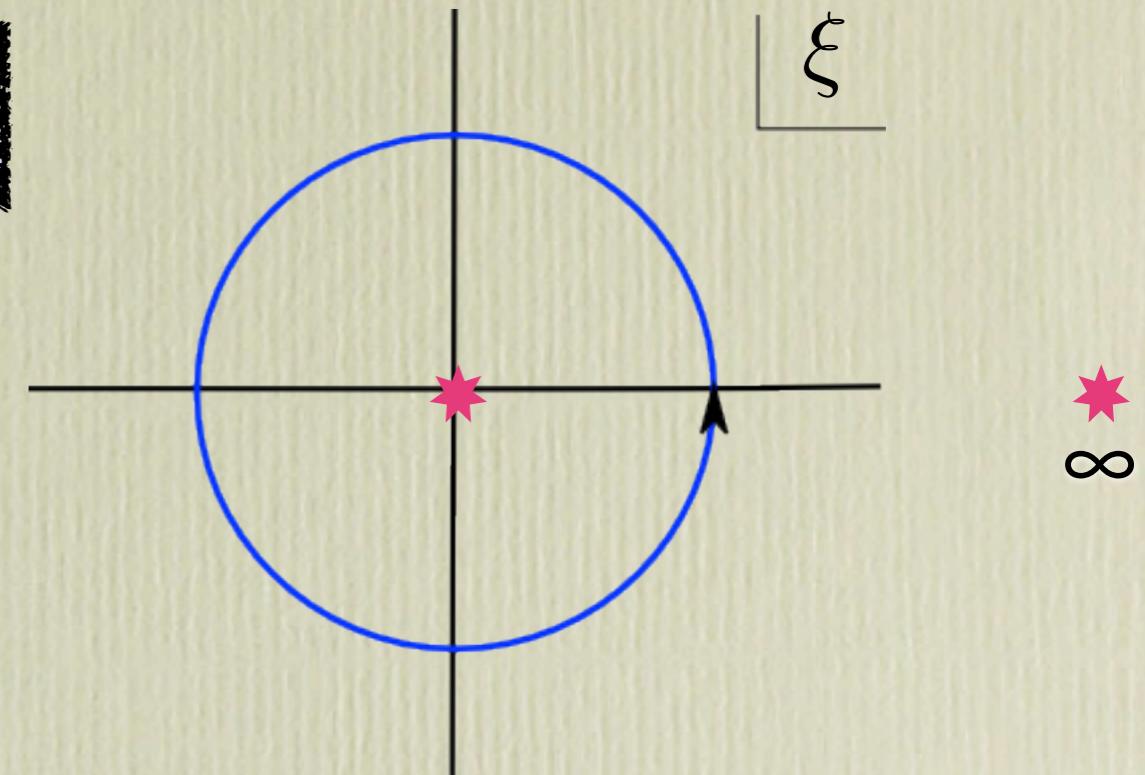


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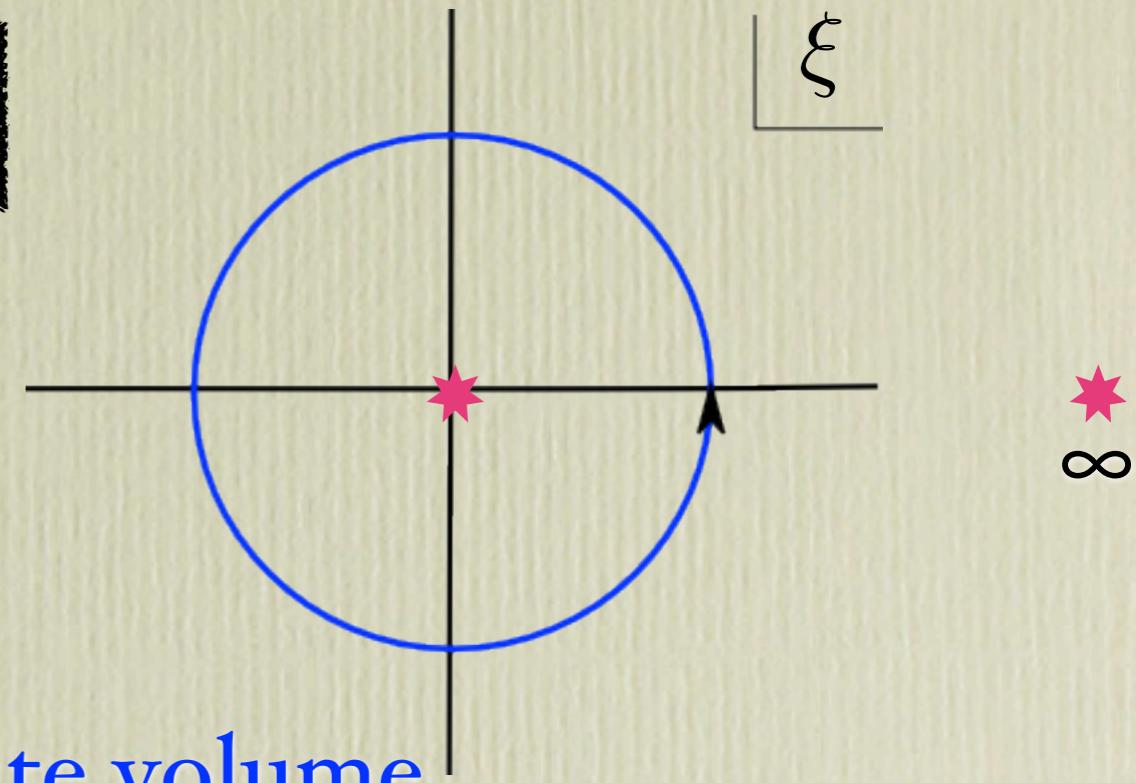
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Yes!

at least for lattice QCD in finite volume



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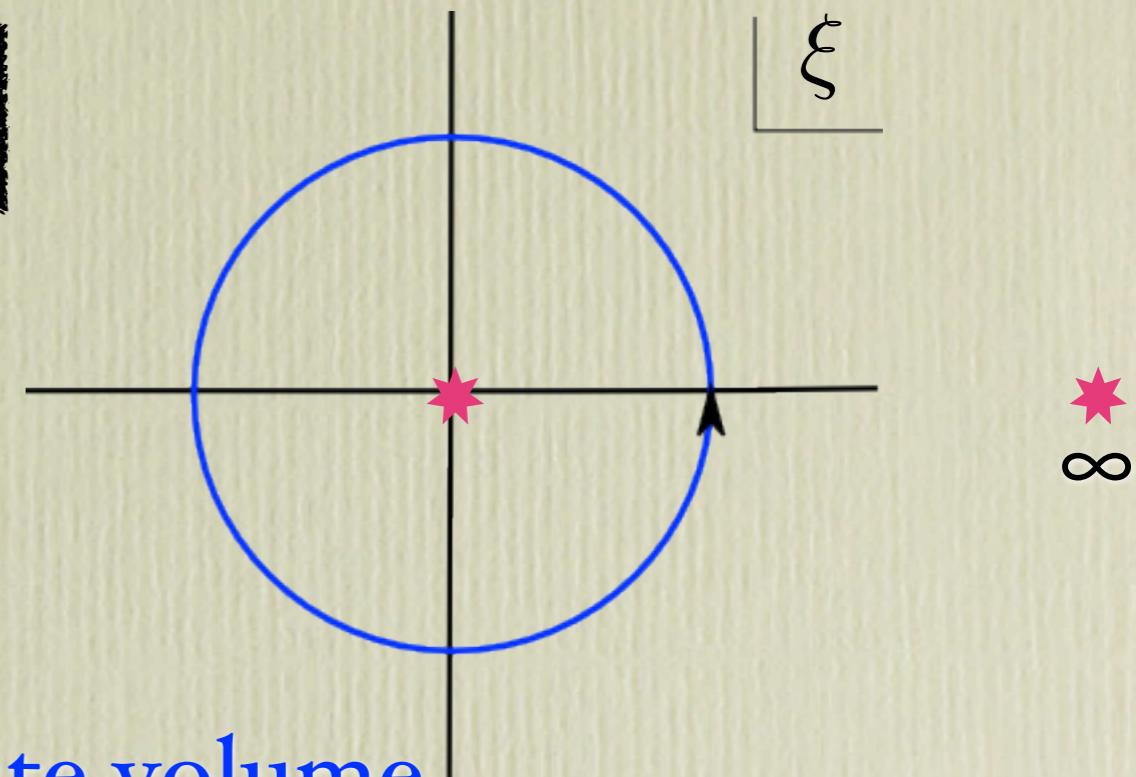
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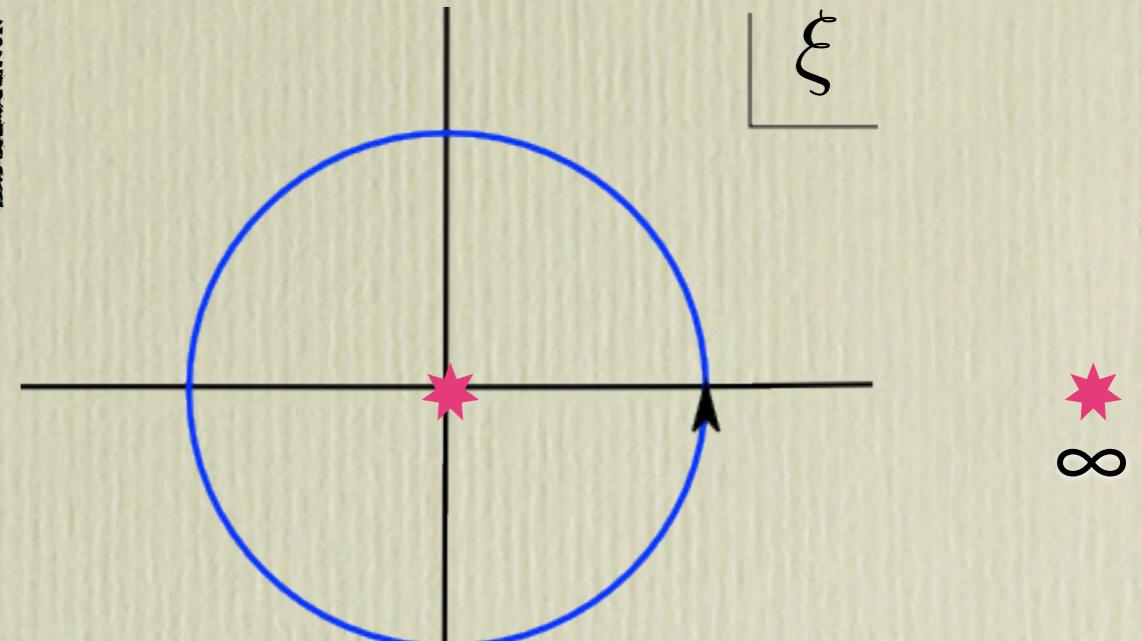
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$$\xi = e^{\frac{\mu}{T}} = 0, \infty$$

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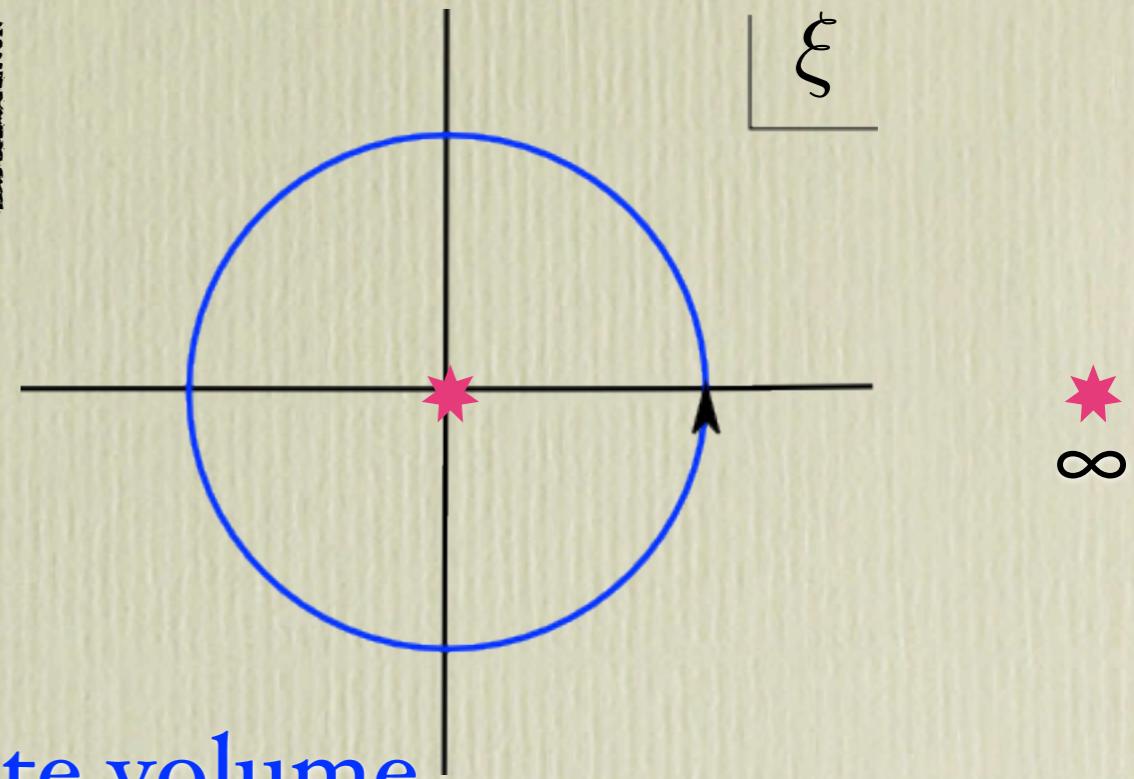
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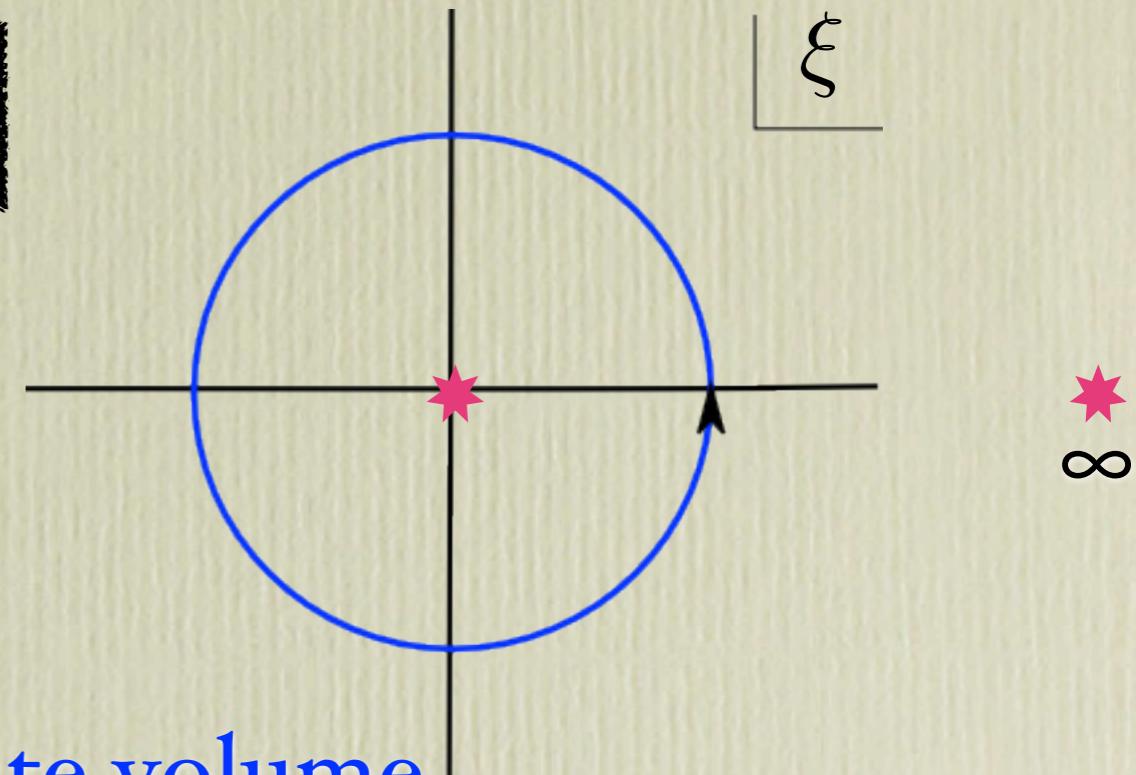
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Phase transition is related to zeros of $Z_G(\xi)$



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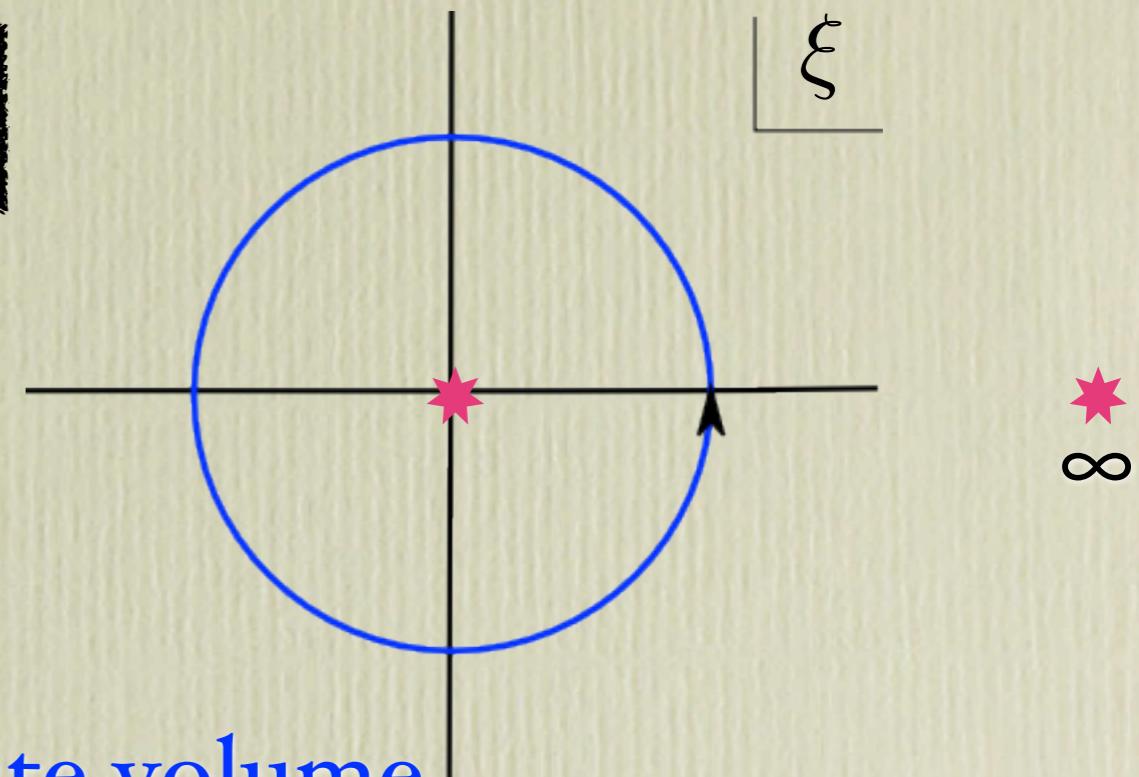
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Phase transition is related to zeros of $Z_G(\xi)$

Lee-Yang zeros!



Introduction

Analytic continuation is perfectly safe for $Z_G(\xi)$!

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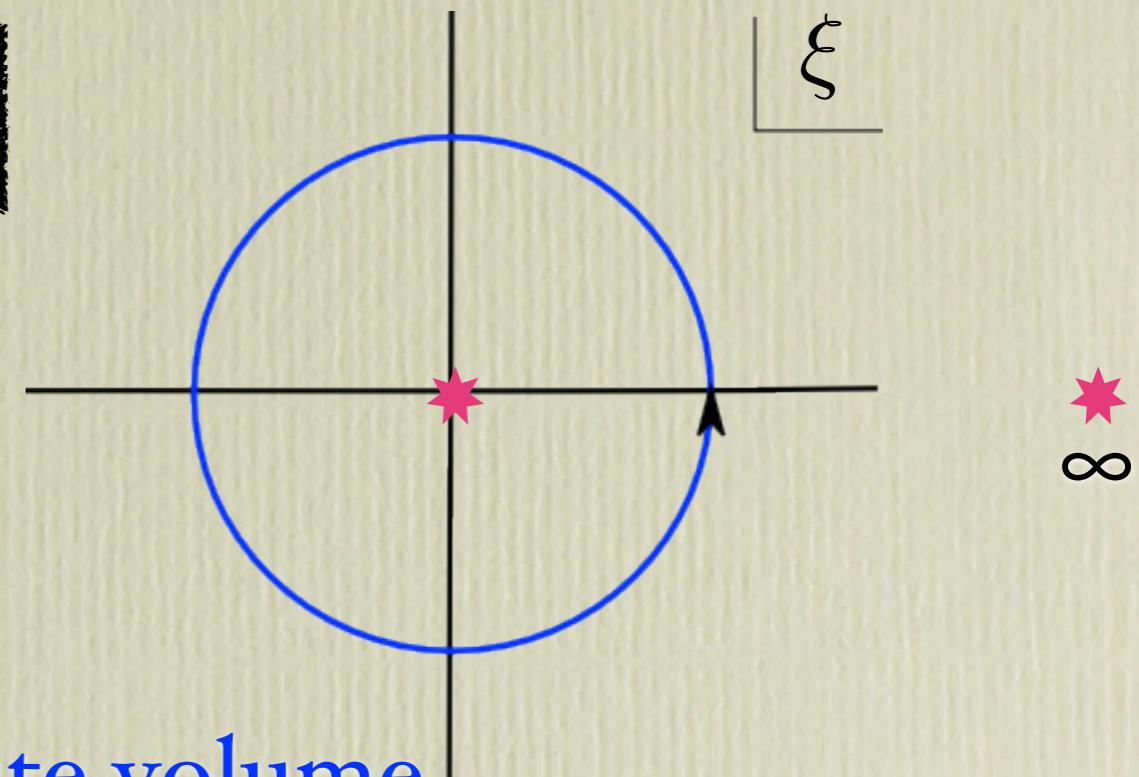
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Plan of the talk

- ✓ 1. Introduction
- 2. Winding number expansion
- 3. Numerical setup
- 4. Numerical results
- 5. Hadronic observables
- 6. Conclusion

Hopping parameter expansion

Everyone need to evaluated $\text{Det } D(\mu)$ to get $Z_G(\mu)$

Direct evaluation is still expensive...

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Direct evaluation is still expensive...

We want a cheaper method!

Hopping parameter expansion

Everyone need to evaluated Det D(μ) to get $Z_G(\mu)$

Chemical potential appears in temporal hopping

$$D_W(\mu) = 1 - \kappa Q_s + \kappa e^{\mu a} Q_4^+ + \kappa e^{-\mu a} Q_4^-$$

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easy to expand

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numerically stable

Meng et al. (Kentucky)

Winding number expansion

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 non-trivial μ dependence

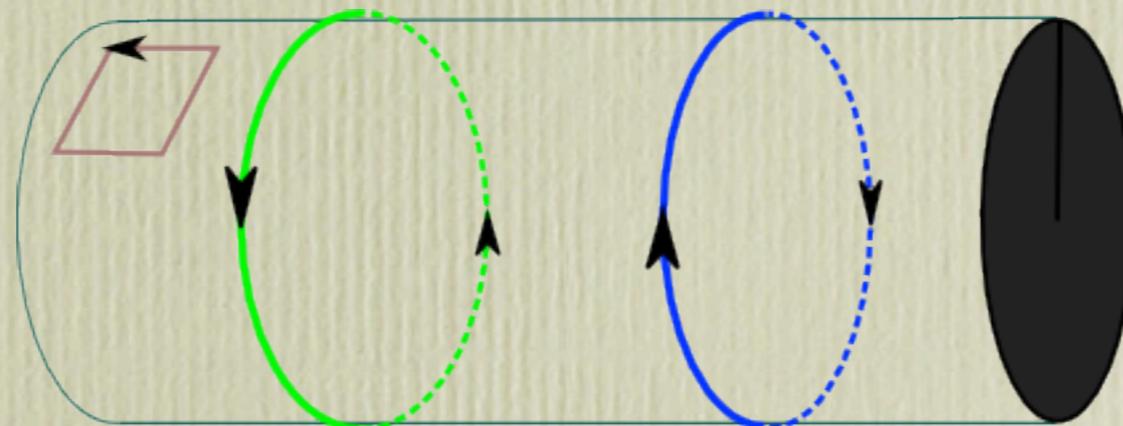
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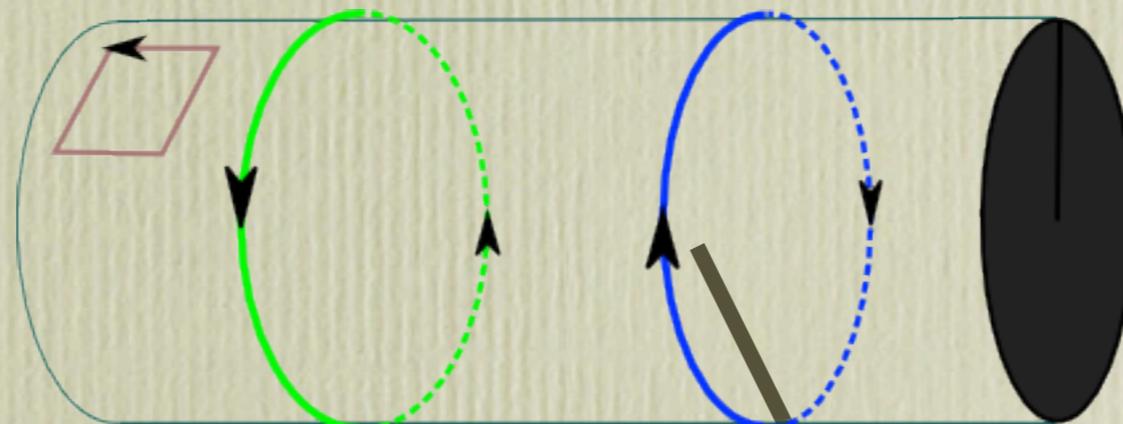
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$$(e^{\mu a})^{N_t} = e^{\mu a N_t} = e^{\mu/T} = \xi$$

Winding number expansion

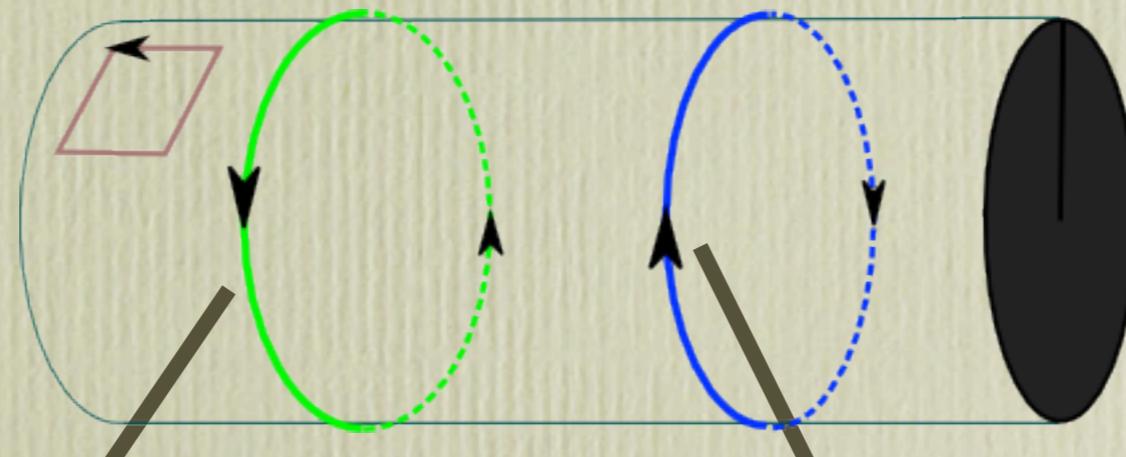
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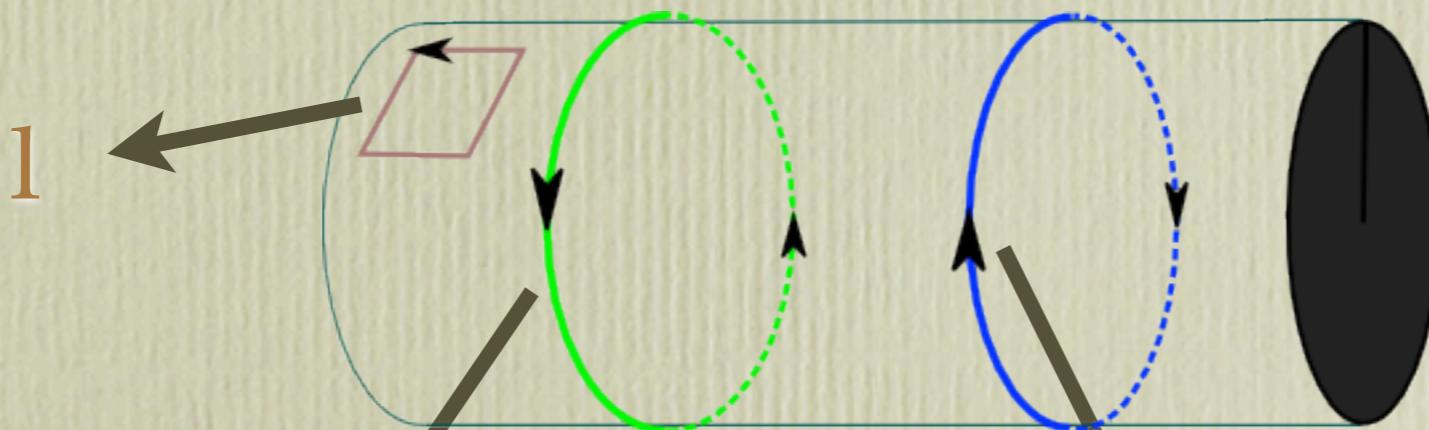
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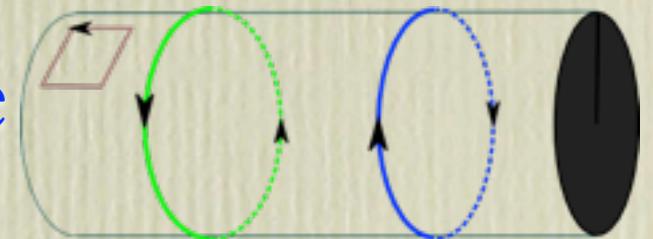
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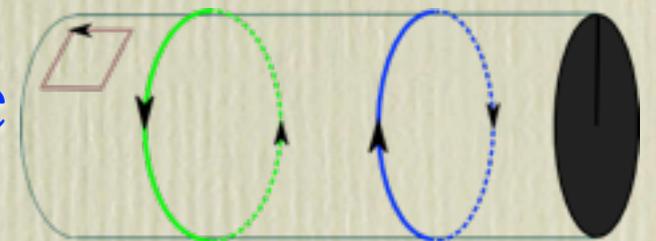
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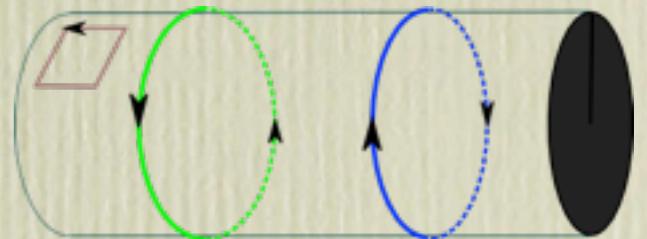
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resummation

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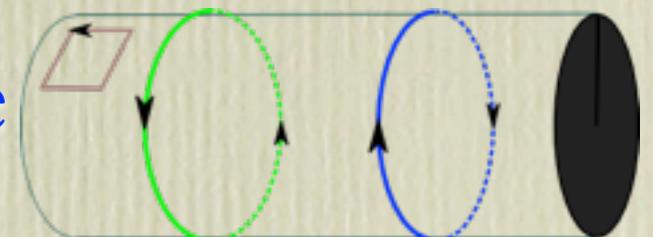
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$$= \sum_{N=-\infty}^{\infty} W_N \xi^N$$

Kentucky '08

Evaluation of $Z_c(n)$

Kentucky '08

Grand partition function $Z_G(\mu) \leftarrow$ re-weighting technique

Evaluation of Zc(n)

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fugacity exp. $= \left\langle \frac{\text{Det} D_W(\mu)}{\text{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$ set to 0 or imaginary winding number exp.

$$\sum_{n=-\infty}^{\infty} Z_C(n) \xi^n = \left\langle \frac{\exp \left(\sum_{k=-\infty}^{\infty} W_k \xi^k \right)}{\text{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$$

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Kentucky '08

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$$Z_G(\mu) = \int DU \frac{\text{Det} D_W(\mu)}{\text{Det} D_W(\mu_0)} \text{Det} D_W(\mu_0) e^{-S_G}$$

fugacity exp. $= \left\langle \frac{\text{Det} D_W(\mu)}{\text{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$ set to 0 or imaginary winding number exp.

$$\sum_{n=-\infty}^{\infty} Z_C(n) \xi^n = \left\langle \frac{\exp \left(\sum_{k=-\infty}^{\infty} W_k \xi^k \right)}{\text{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$$

Evaluation of Zc(n)

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$Z_C(n) \xi^n$ is circled in red.

Cauchy's integral theorem

$$Z_C(n) = \oint \frac{d\xi}{2\pi i} \xi^{-n-1} \left\langle \frac{\exp \left(\sum_{k=-\infty}^{\infty} W_k \xi^k \right)}{\text{Det} D_W(\mu_0)} \right\rangle_0$$

Evaluation of Zc(n)

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fugacity exp.

winding number exp.

Fourier transformation

$$Z_C(n) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} \left\langle \frac{\exp \left(\sum_{k=-\infty}^{\infty} W_k e^{ik\theta} \right)}{\text{Det} D_W(\mu_0)} \right\rangle_0$$

Plan of the talk

- ✓ 1. Introduction
- ✓ 2. Winding number expansion
- 3. Numerical setup
- 4. Numerical results
- 5. Hadronic observables
- 6. Conclusion

Numerical setup

- ★ Iwasaki gauge action
- ★ Clover fermion $N_f=2$
- APE stout smeared gauge link $c_{SW} = 1.1$
- ★ Box sizes $8^3 \times 4$ $12^3 \times 4$ $16^3 \times 4$

Numerical setup

- ★ Iwasaki gauge action
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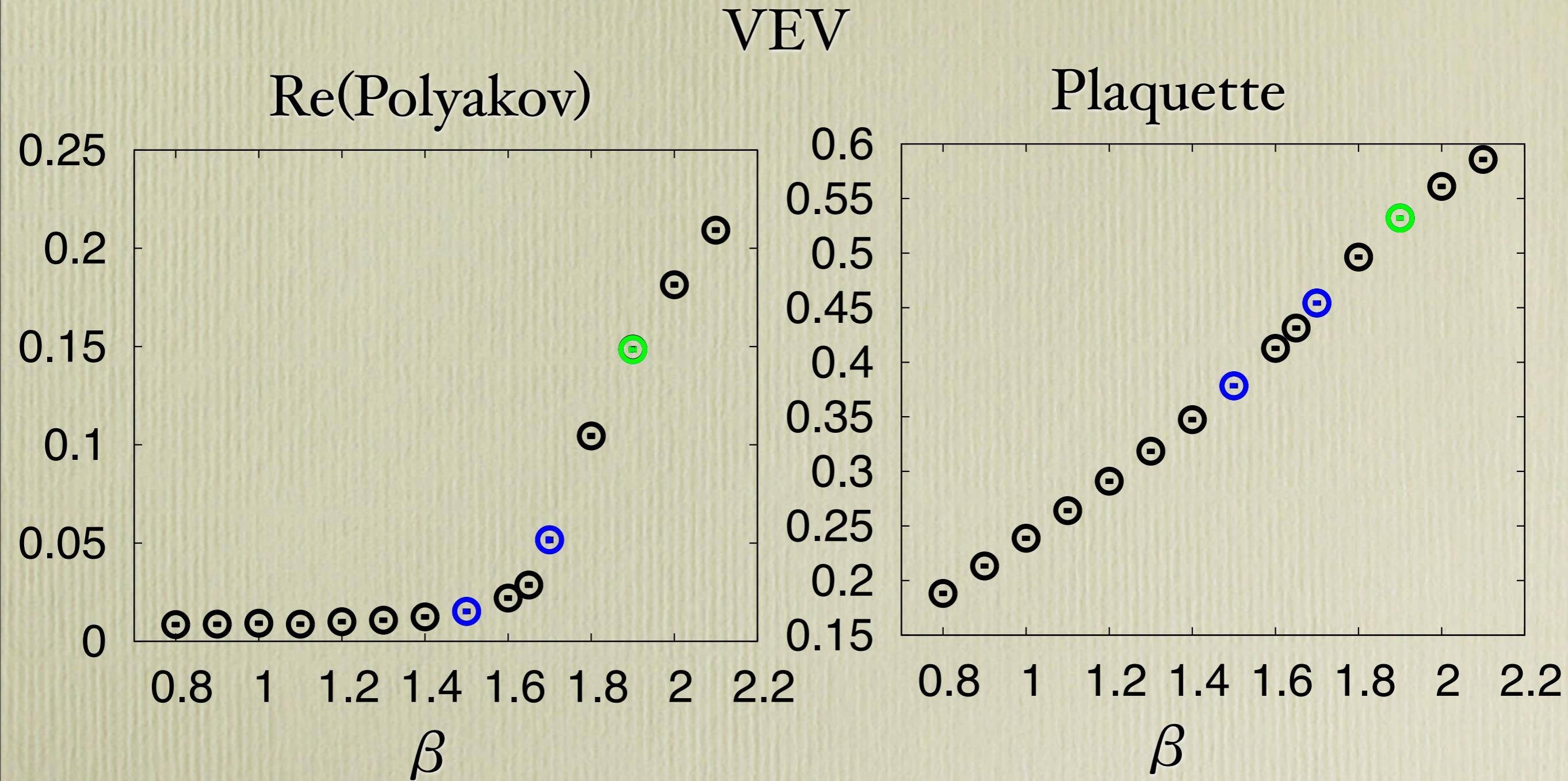
- APE stout smeared gauge link $c_{SW} = 1.1$
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β	κ	PCAC mass
0.9	0.137	0.17(13)
1.1	0.133	0.18(19)
1.3	0.133	0.088(53)
1.5	0.131	0.116(39)
1.7	0.129	0.168(21)
1.9	0.125	0.1076(68)
2.1	0.122	0.1259(11)

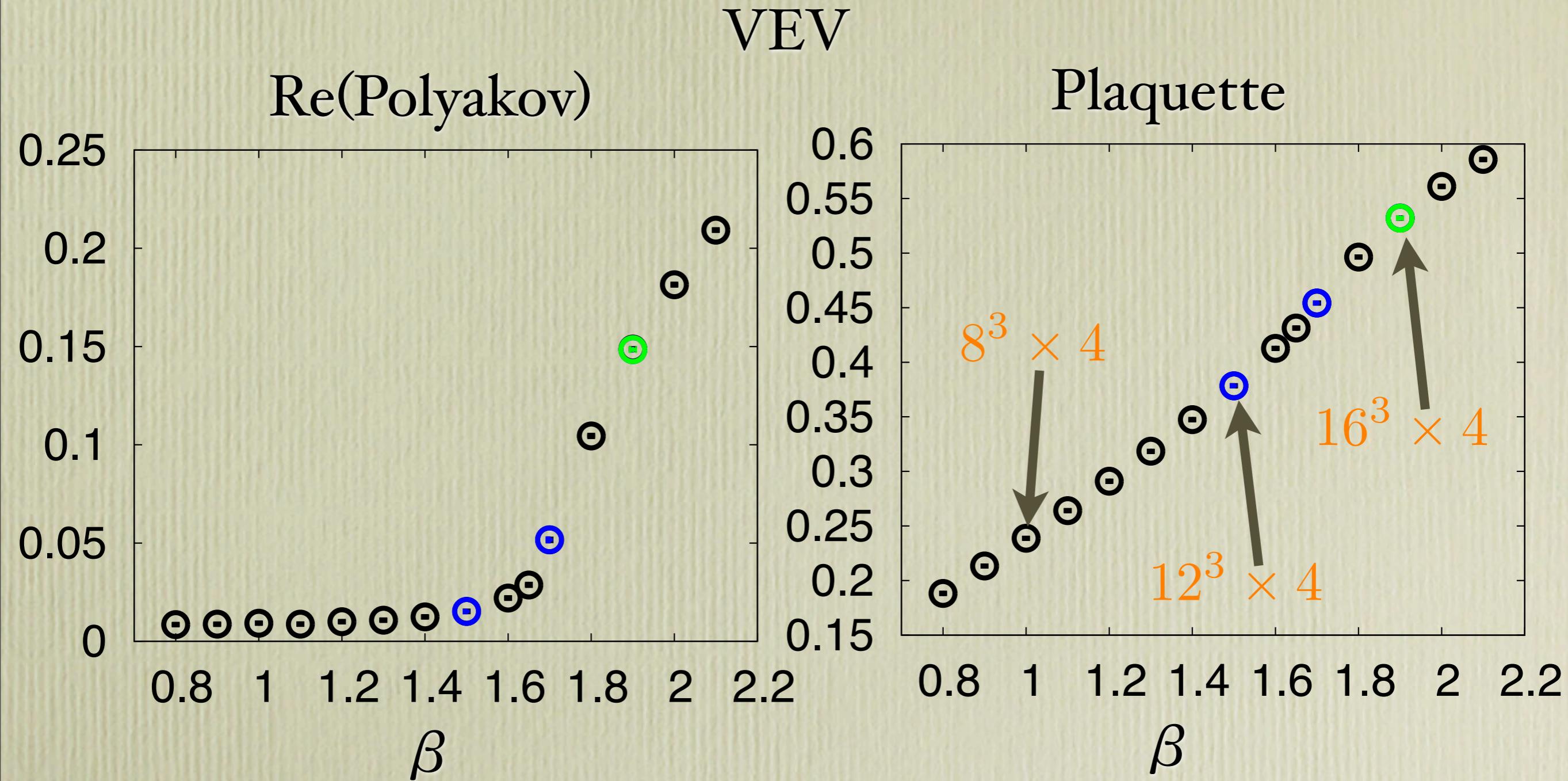
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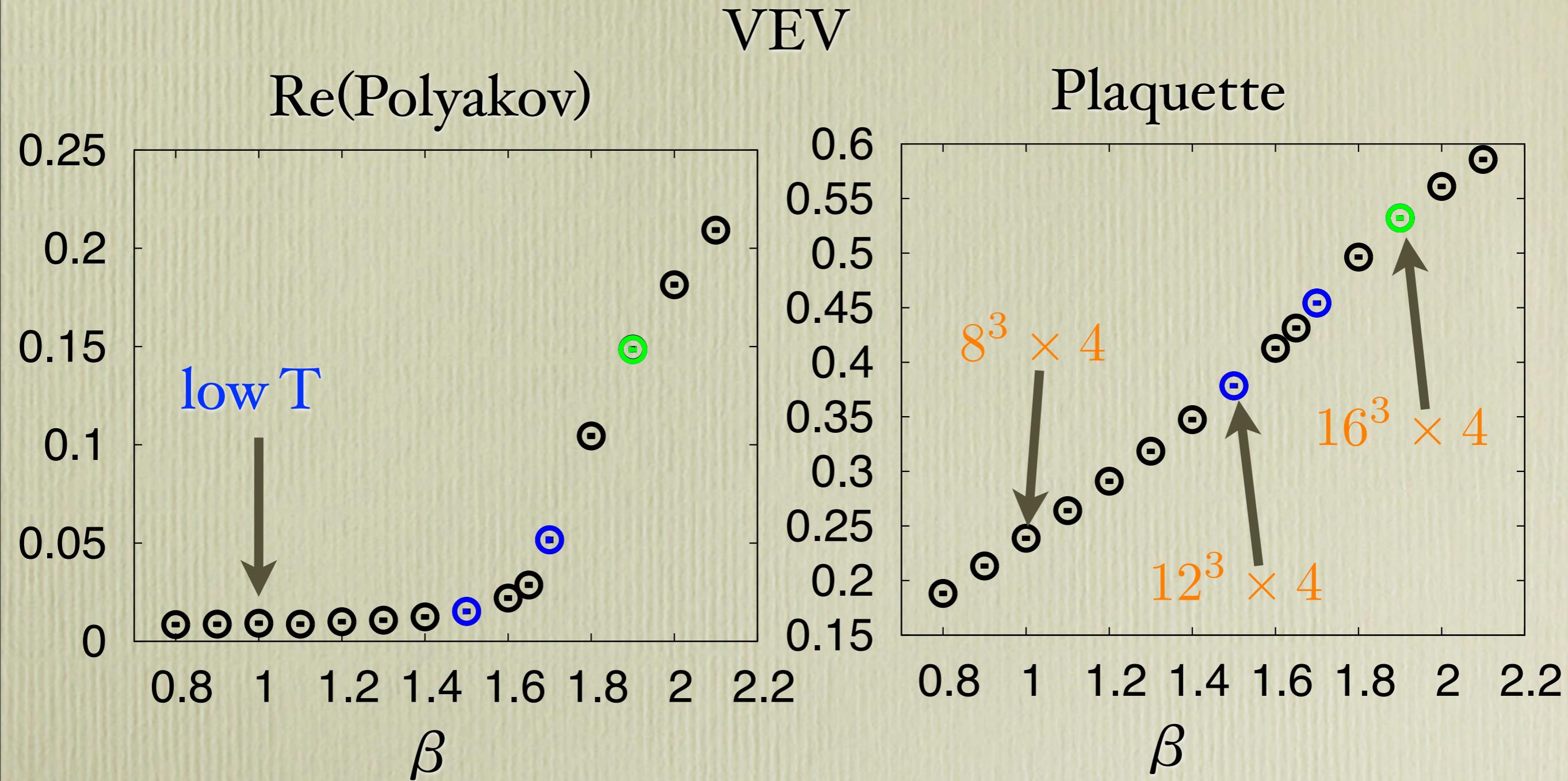
Numerical results (Polyakov loop)



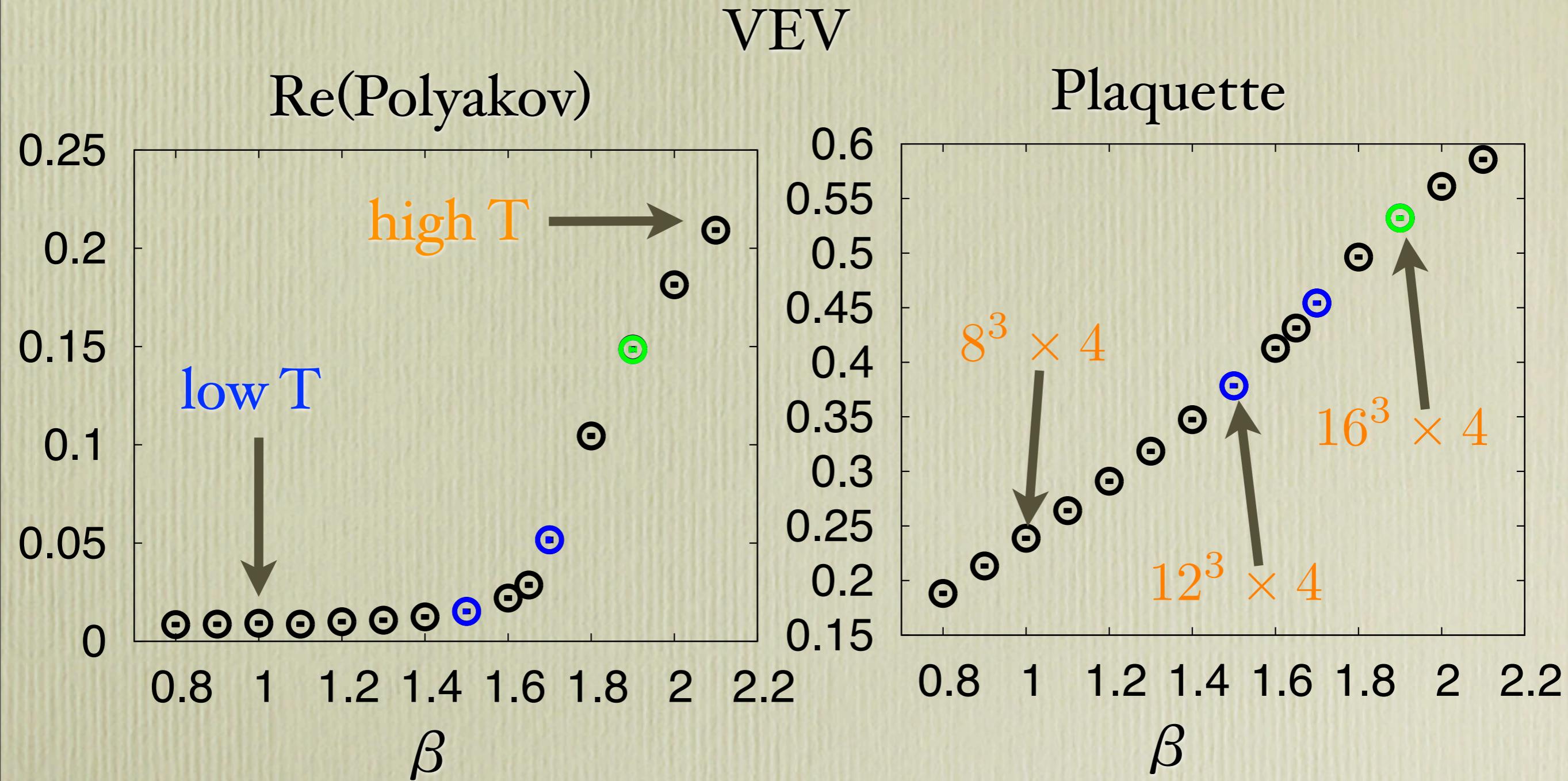
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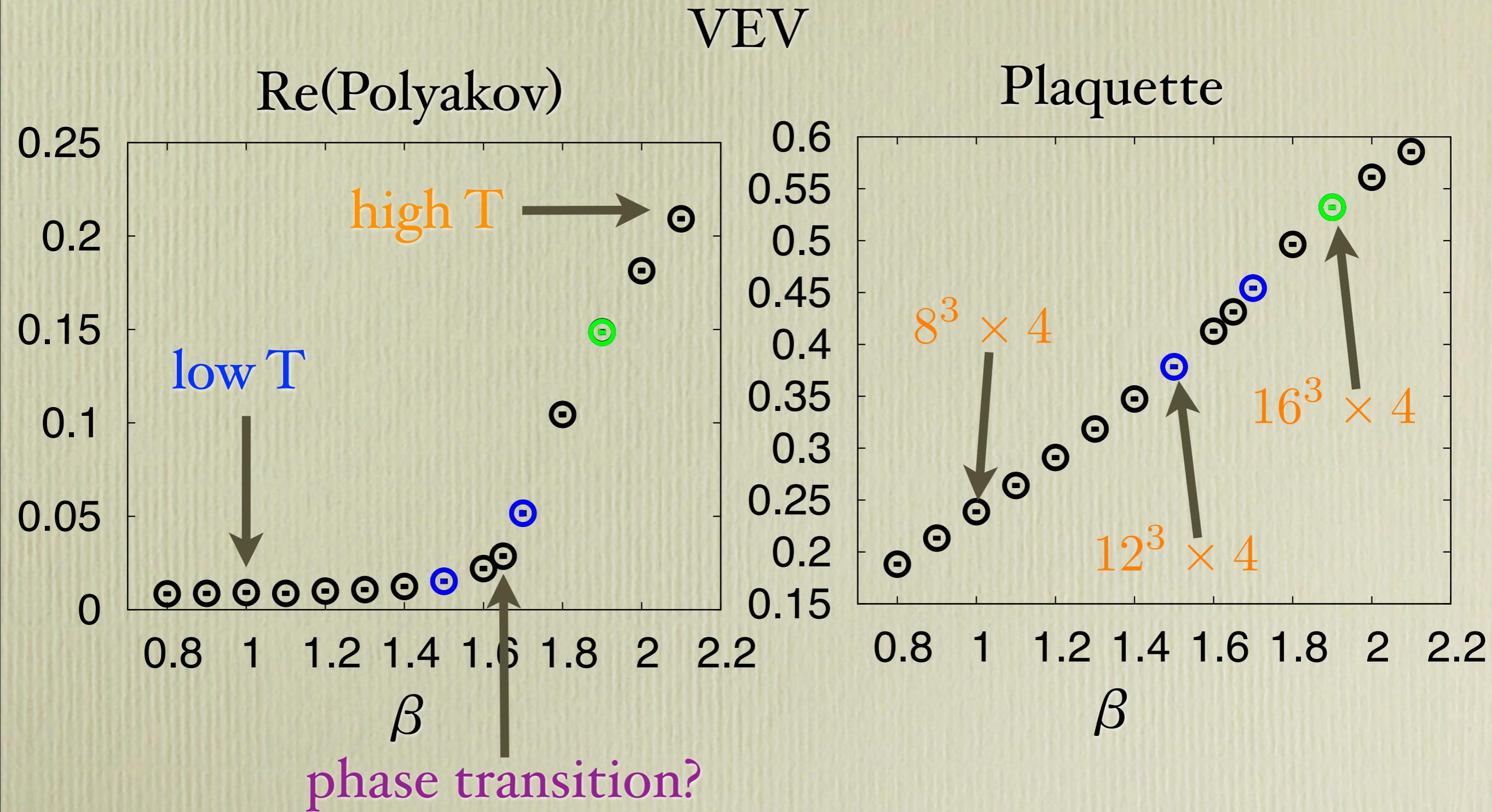
Numerical results (Polyakov loop)



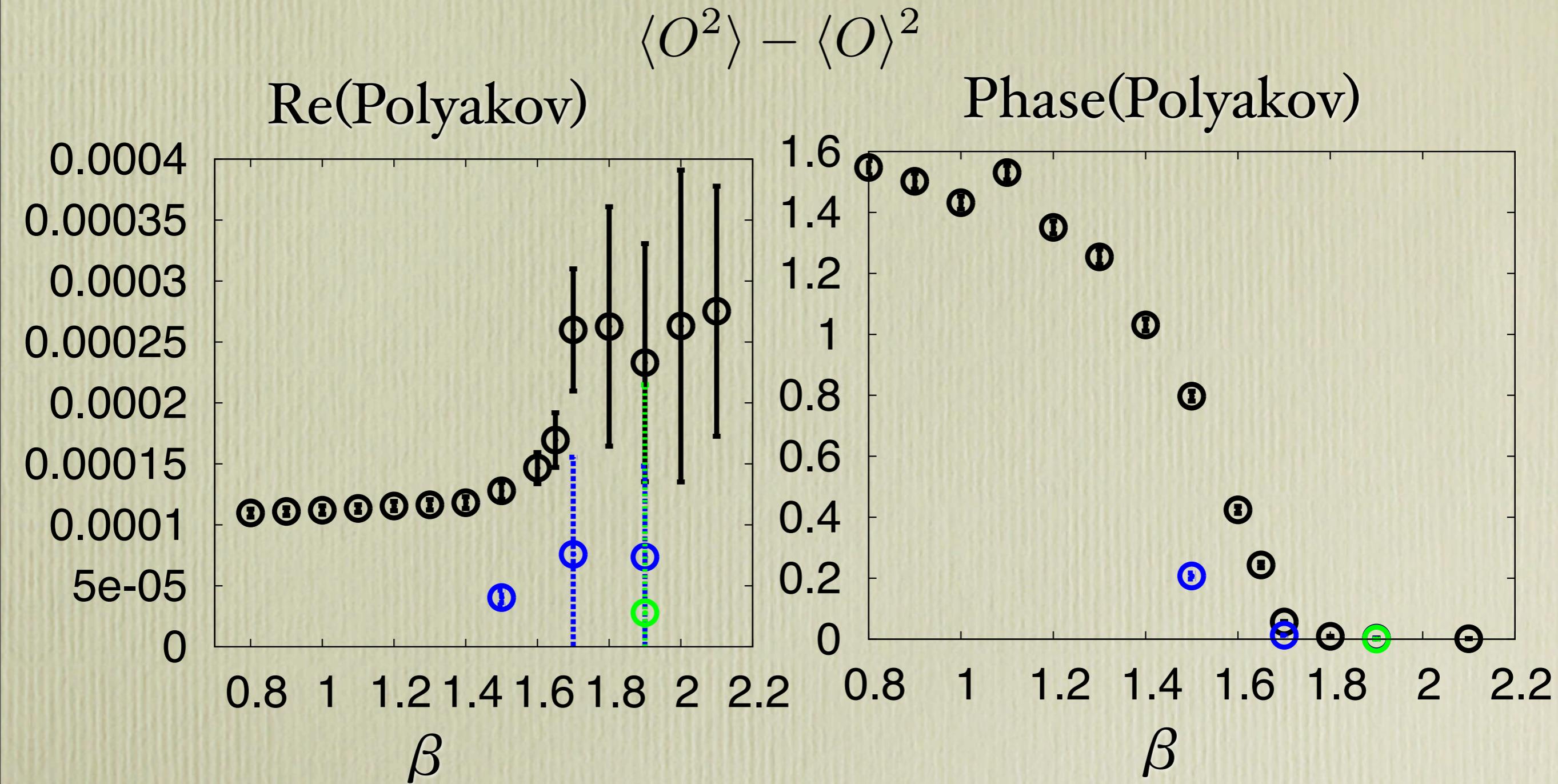
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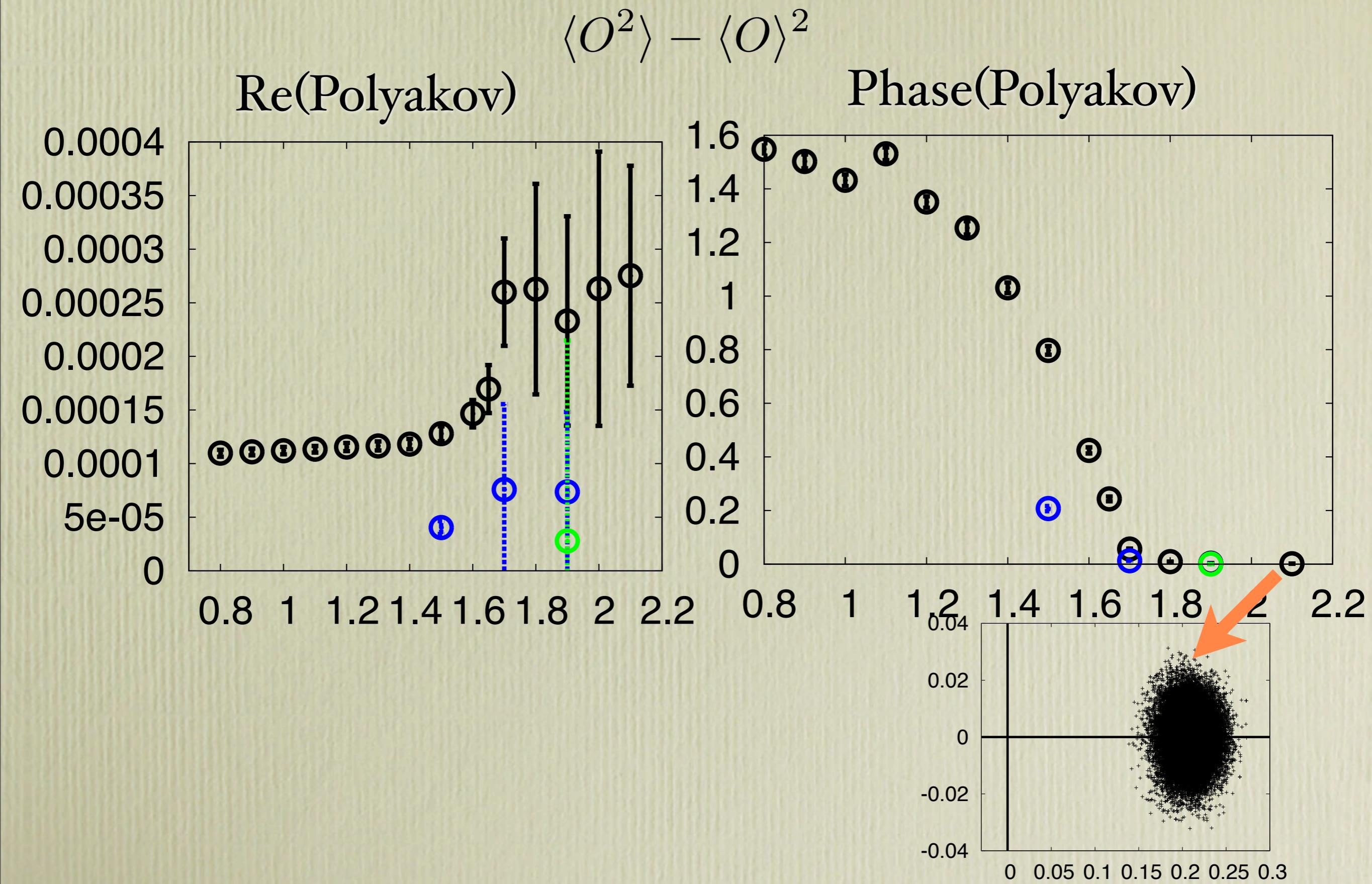
Numerical results (Polyakov loop)



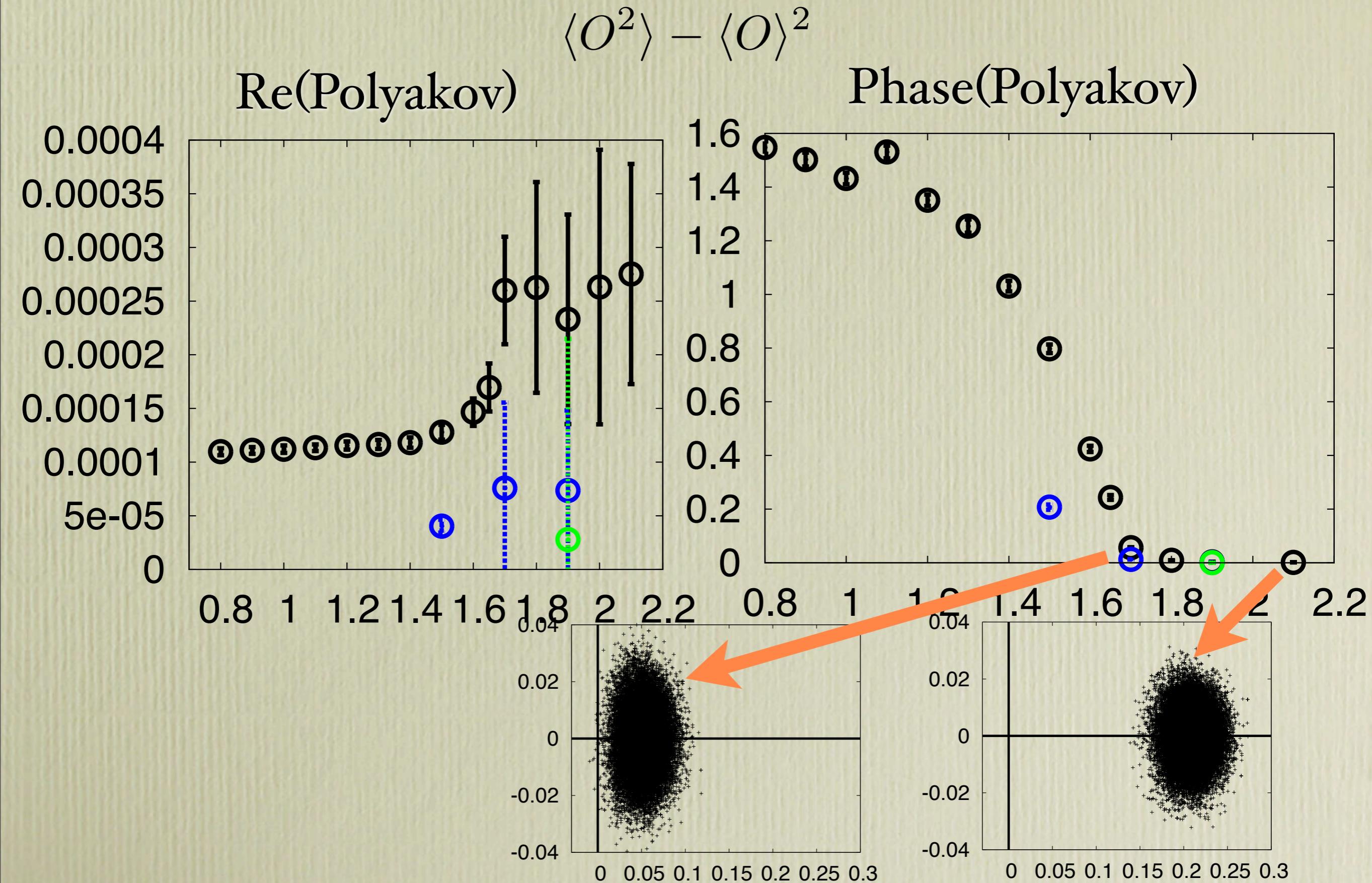
Numerical results (Polyakov loop)



Numerical results (Polyakov loop)



Numerical results (Polyakov loop)

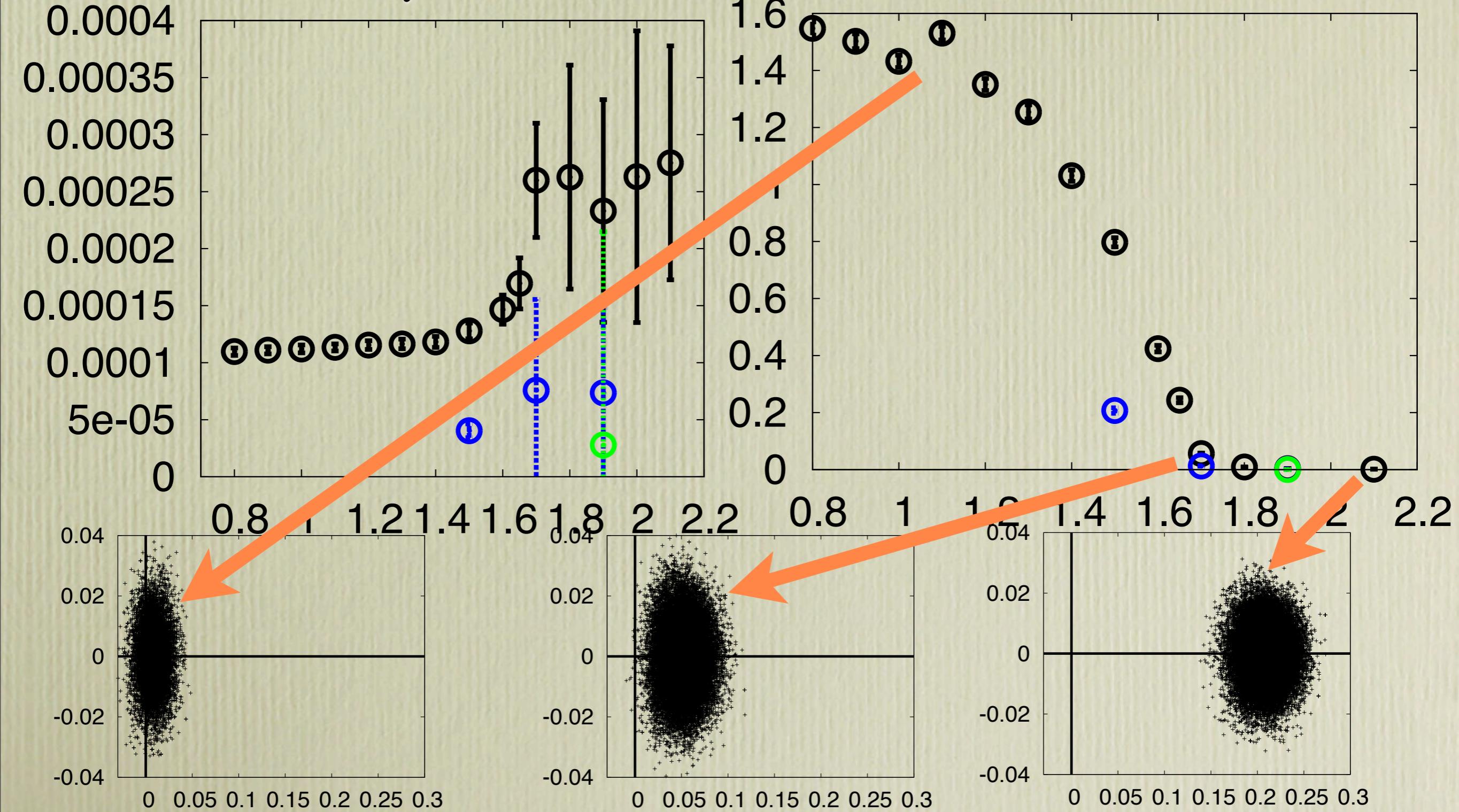


Numerical results (Polyakov loop)

$\langle O^2 \rangle - \langle O \rangle^2$

Re(Polyakov)

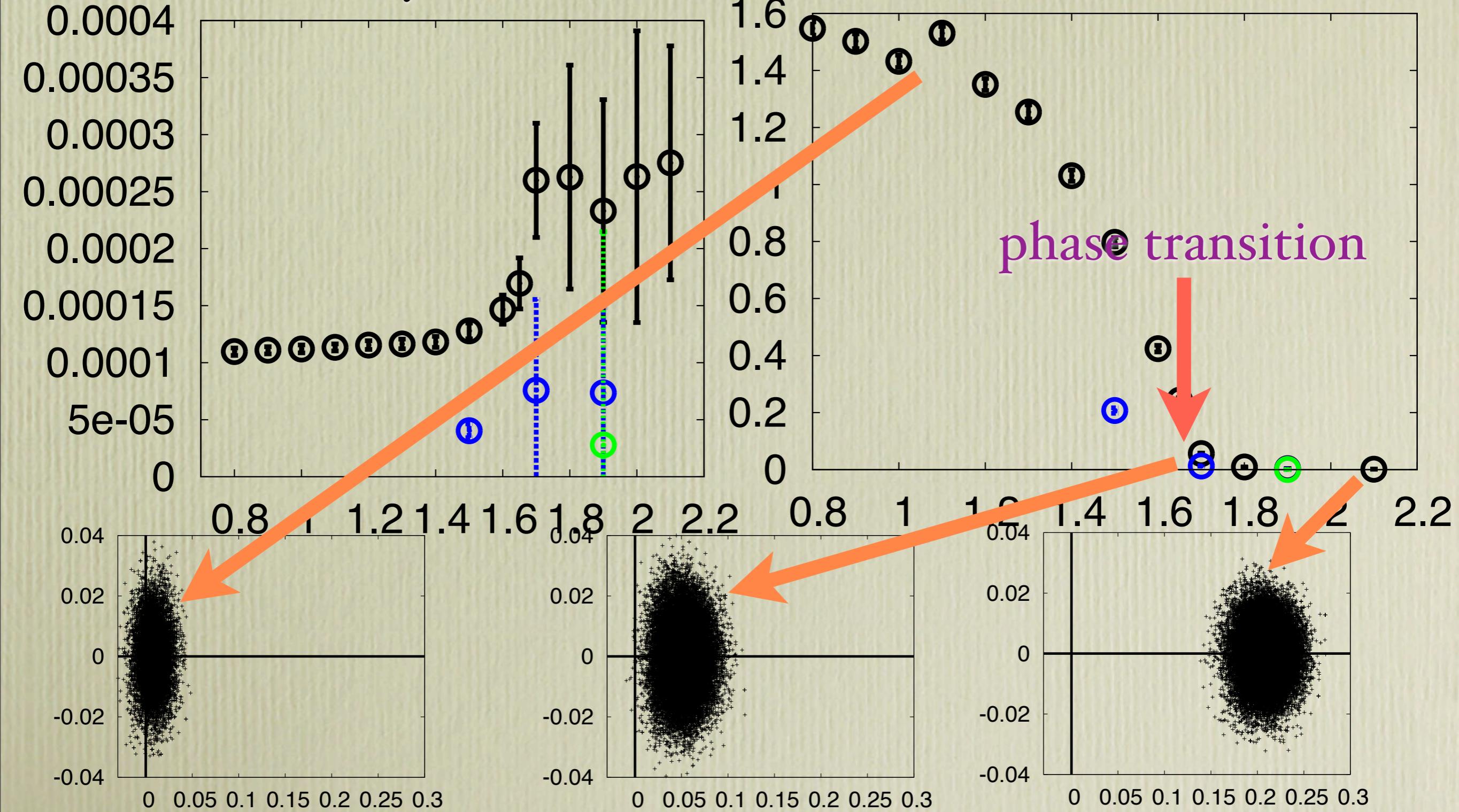
Phase(Polyakov)



Numerical results (Polyakov loop)

$$\langle O^2 \rangle - \langle O \rangle^2$$

Re(Polyakov) Phase(Polyakov)



Numerical results $Z_c(n)$

Before the main dish...

Numerical results $Z_c(n)$

Test of the hopping parameter expansion

Before the main dish...

Numerical results $Z_c(n)$

Test of the hopping parameter expansion

- Where to truncate the expansion?

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Test of the hopping parameter expansion

- Where to truncate the expansion?

We have a good benchmark!

Fugacity expansion by reduction formula

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Fugacity expansion by reduction formula

P. Gibbs

for staggered

A. Hasenfratz and D. Toussaint

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Test of the hopping parameter expansion

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A. Hasenfratz and D. Toussaint

K. Nagata and A. Nakamura

for Wilson

A. Alexandru and U. Wenger

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$$\text{Det} D_W(\mu)$$

Numerical results $Z_c(n)$

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$$\text{Det} D_W(\mu) = C_0 \xi^{-N_R/2} \text{Det} (\xi + Q)$$

Numerical results $Z_c(n)$

Test of the hopping parameter expansion

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$$\text{Det} D_W(\mu) = C_0 \xi^{-N_R/2} \text{Det} (\xi + Q) = \sum_{n=-\infty}^{\infty} c_n(U_\mu) \xi^n$$

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Exact!

Numerical results $Z_c(n)$

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Exact!

Computational cost is heavy...

Numerical results $Z_c(n)$

Canonical partition function measured on a single conf.

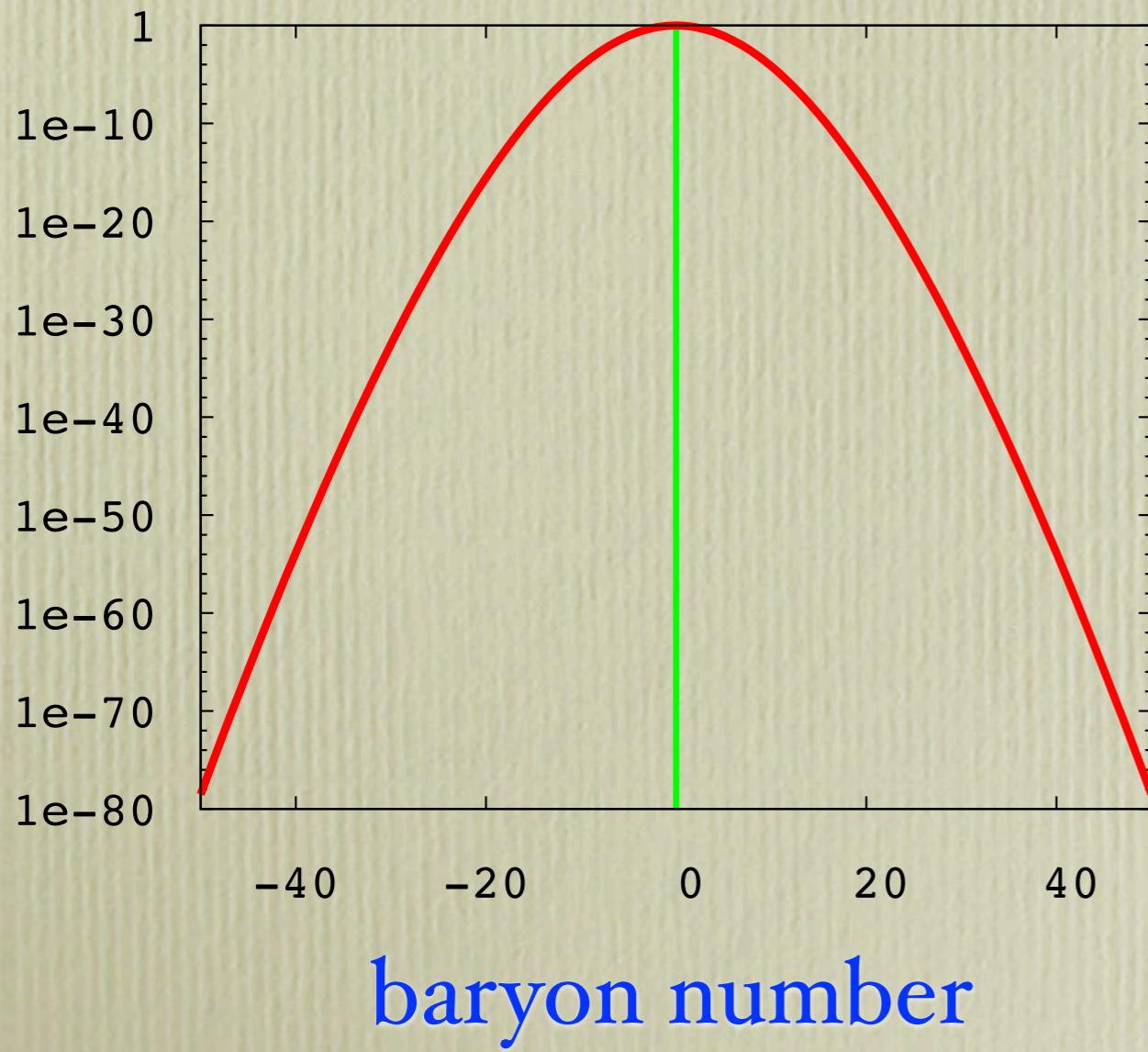
$$8^3 \times 4 \quad \beta = 1.9 \quad \kappa = 0.1250 \quad am_{\text{PCAC}} = 0.1076(68) \quad \mu = 0$$

Numerical results $Z_C(n)$

Canonical partition function measured on a single conf.

$8^3 \times 4$ $\beta = 1.9$ $\kappa = 0.1250$ $am_{\text{PCAC}} = 0.1076(68)$ $\mu = 0$

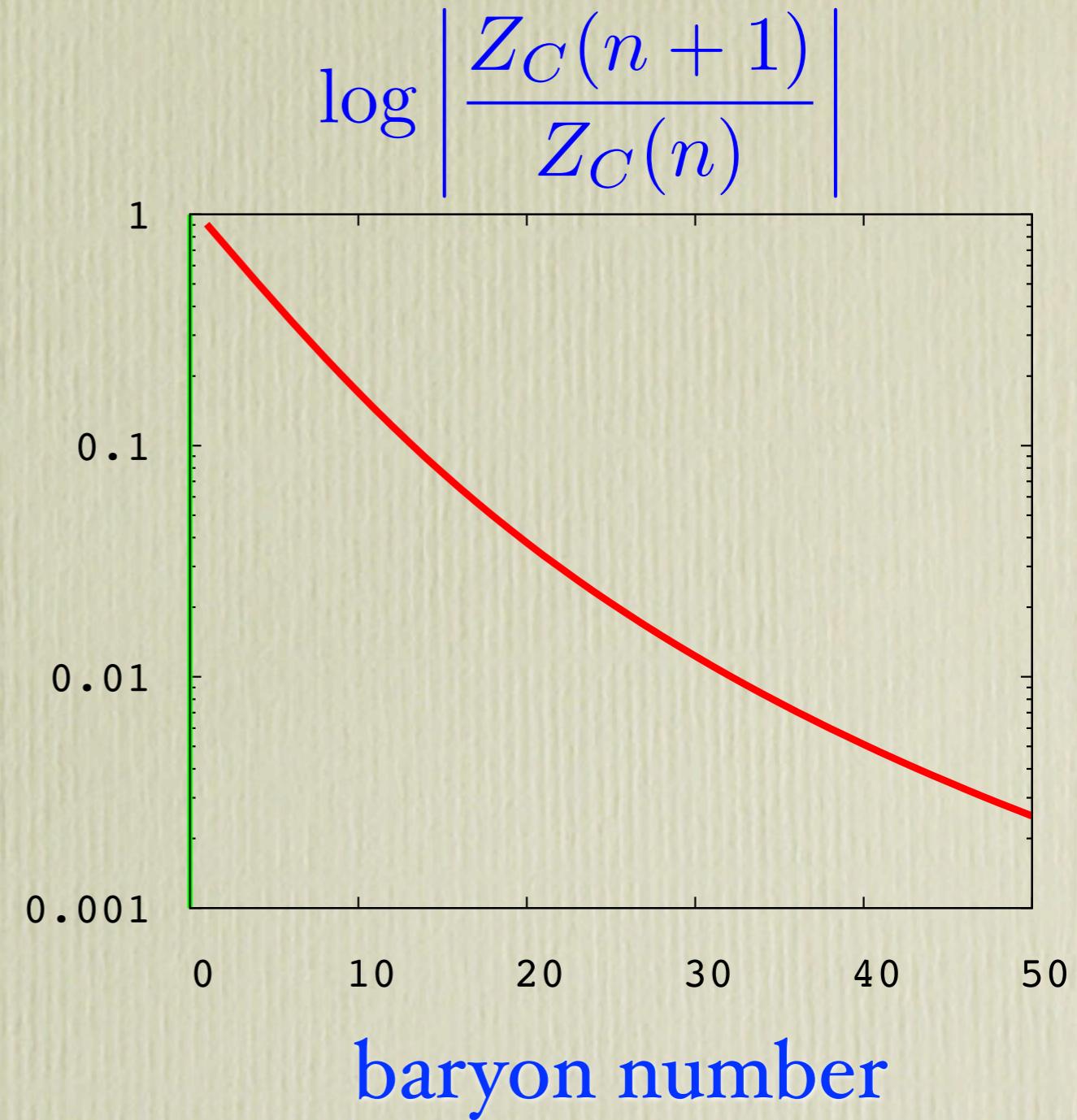
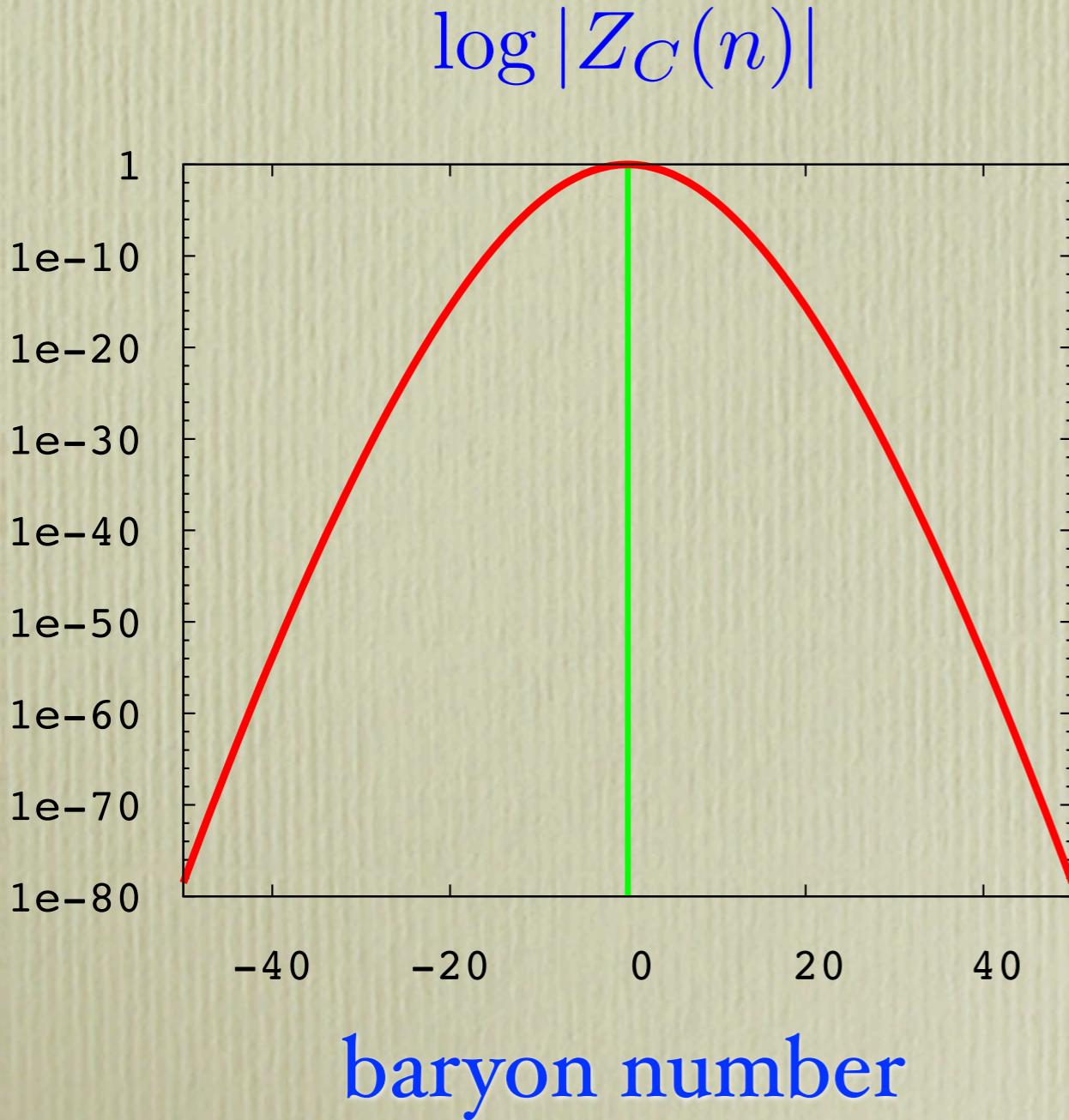
$\log |Z_C(n)|$



Numerical results $Z_C(n)$

Canonical partition function measured on a single conf.

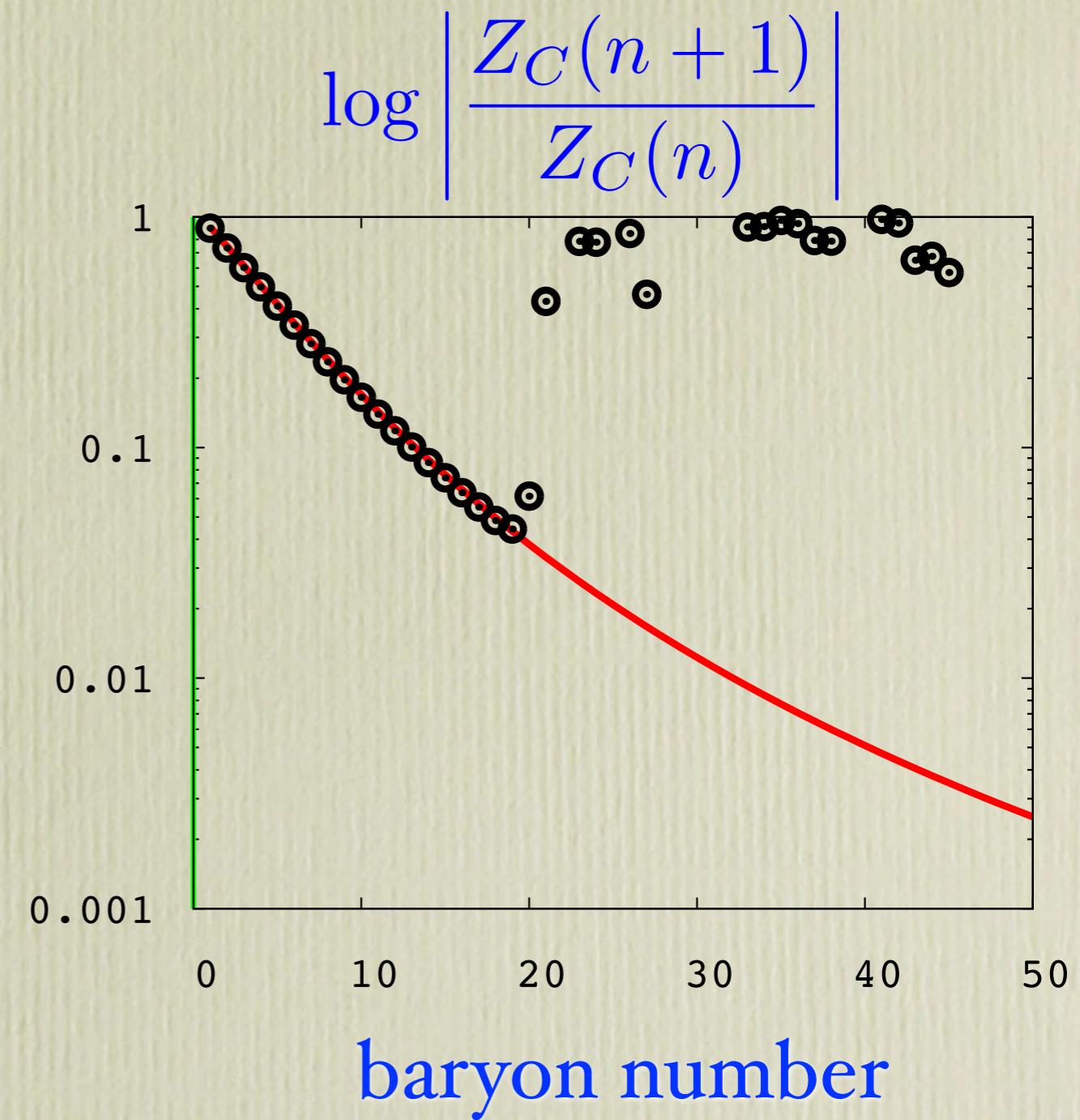
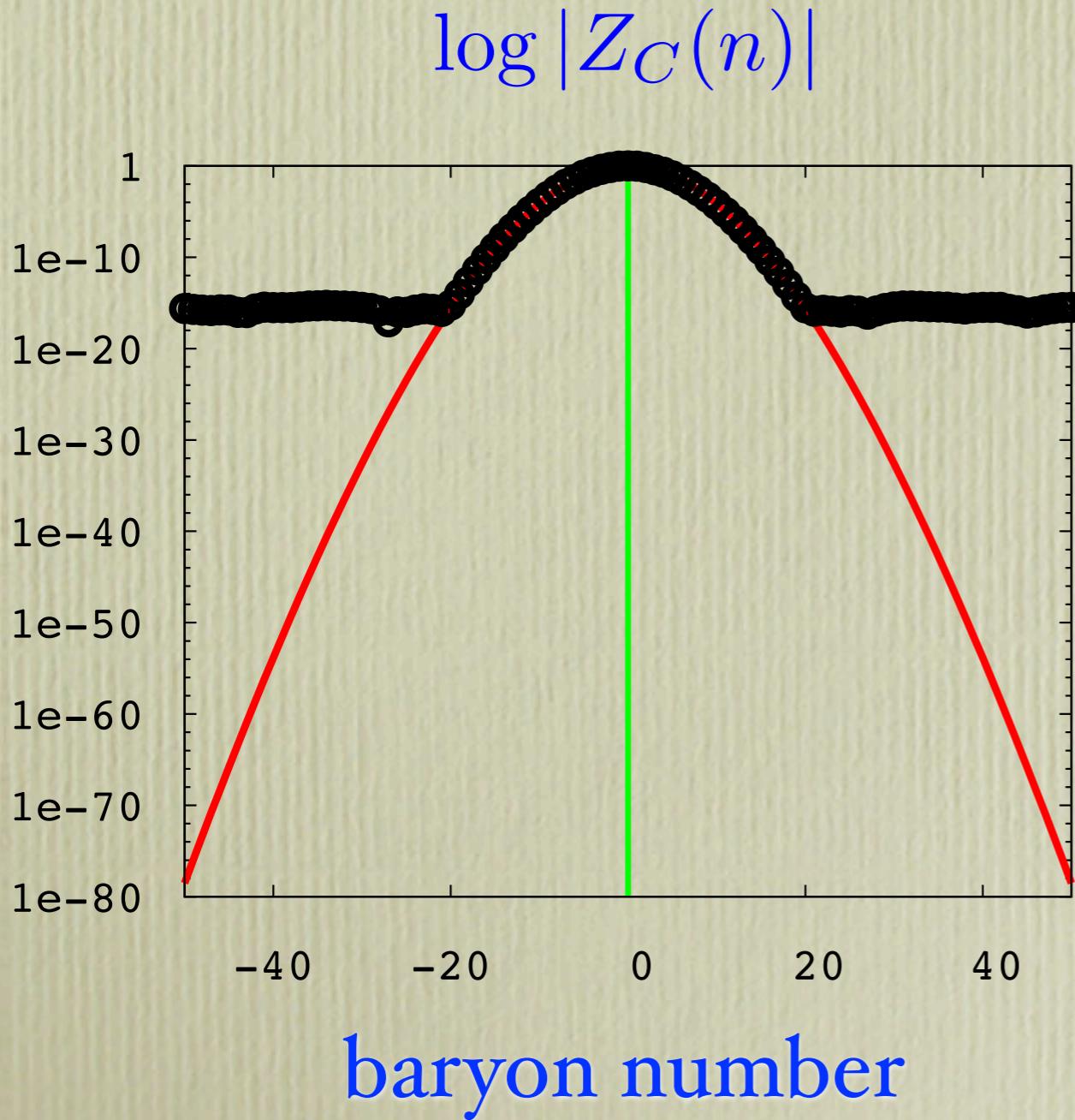
$8^3 \times 4$ $\beta = 1.9$ $\kappa = 0.1250$ $am_{\text{PCAC}} = 0.1076(68)$ $\mu = 0$



Numerical results $Z_C(n)$

Double precision is not enough for Fourier tr!

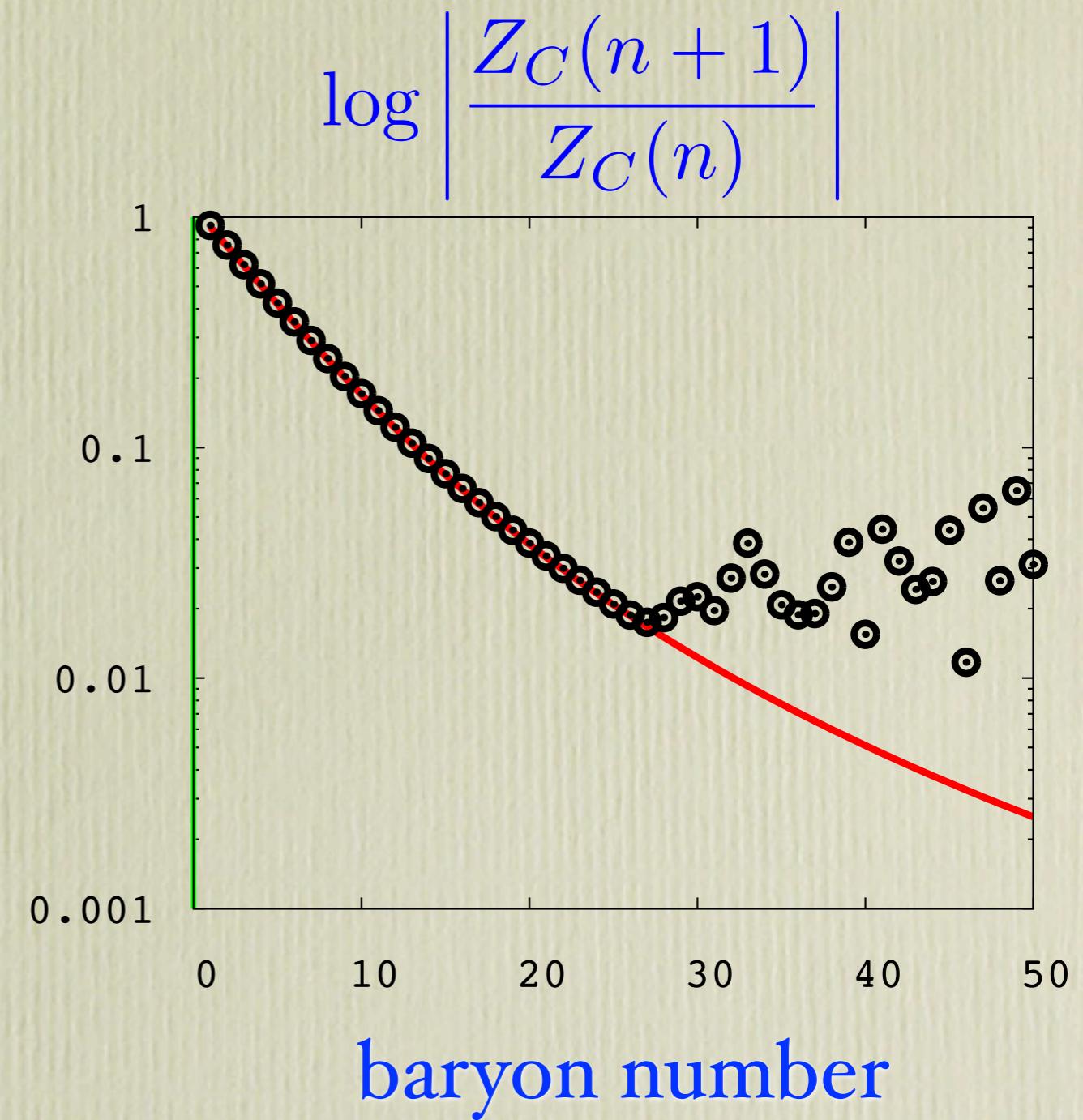
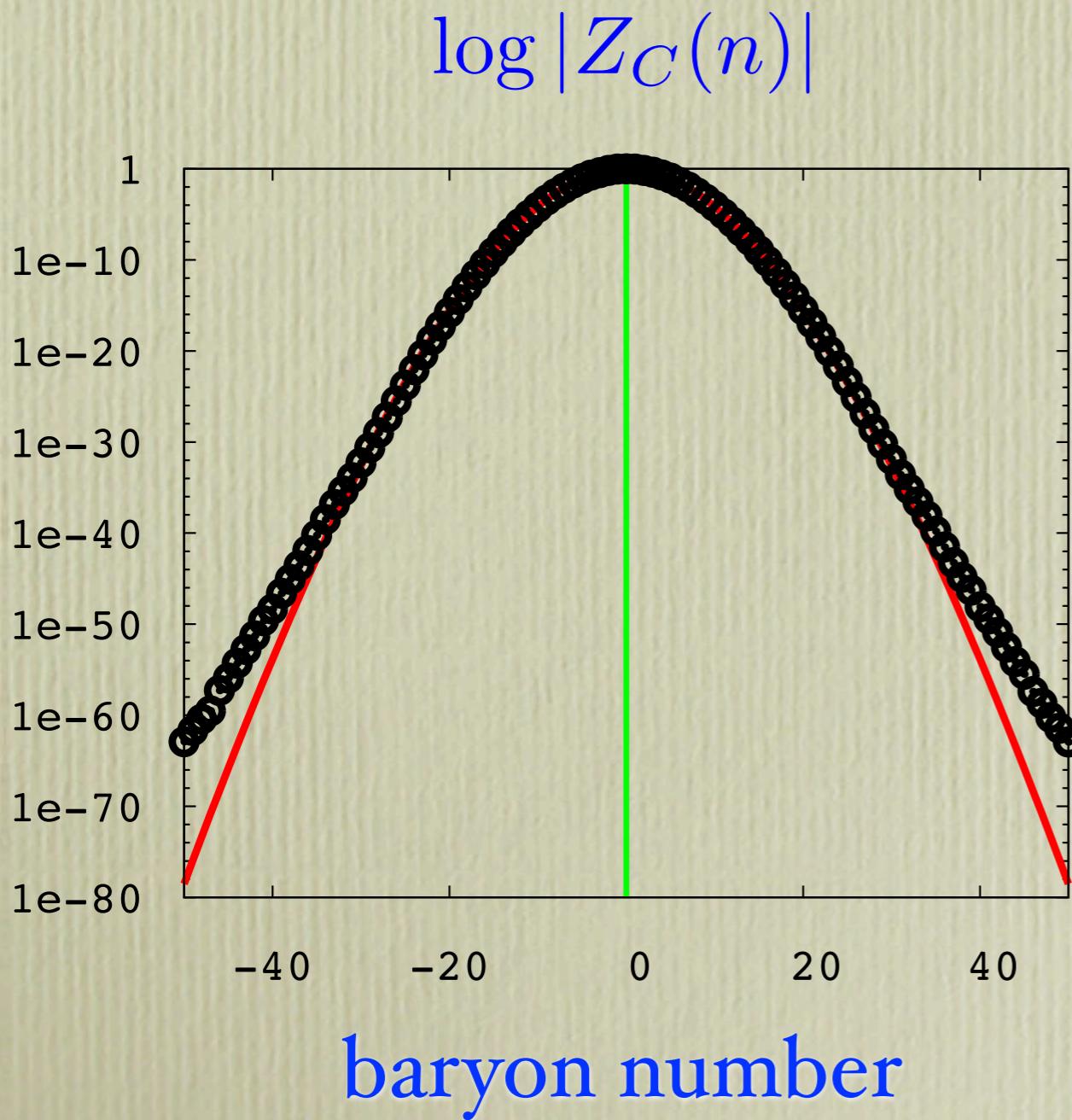
winding number=120 (number of hop =480)



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

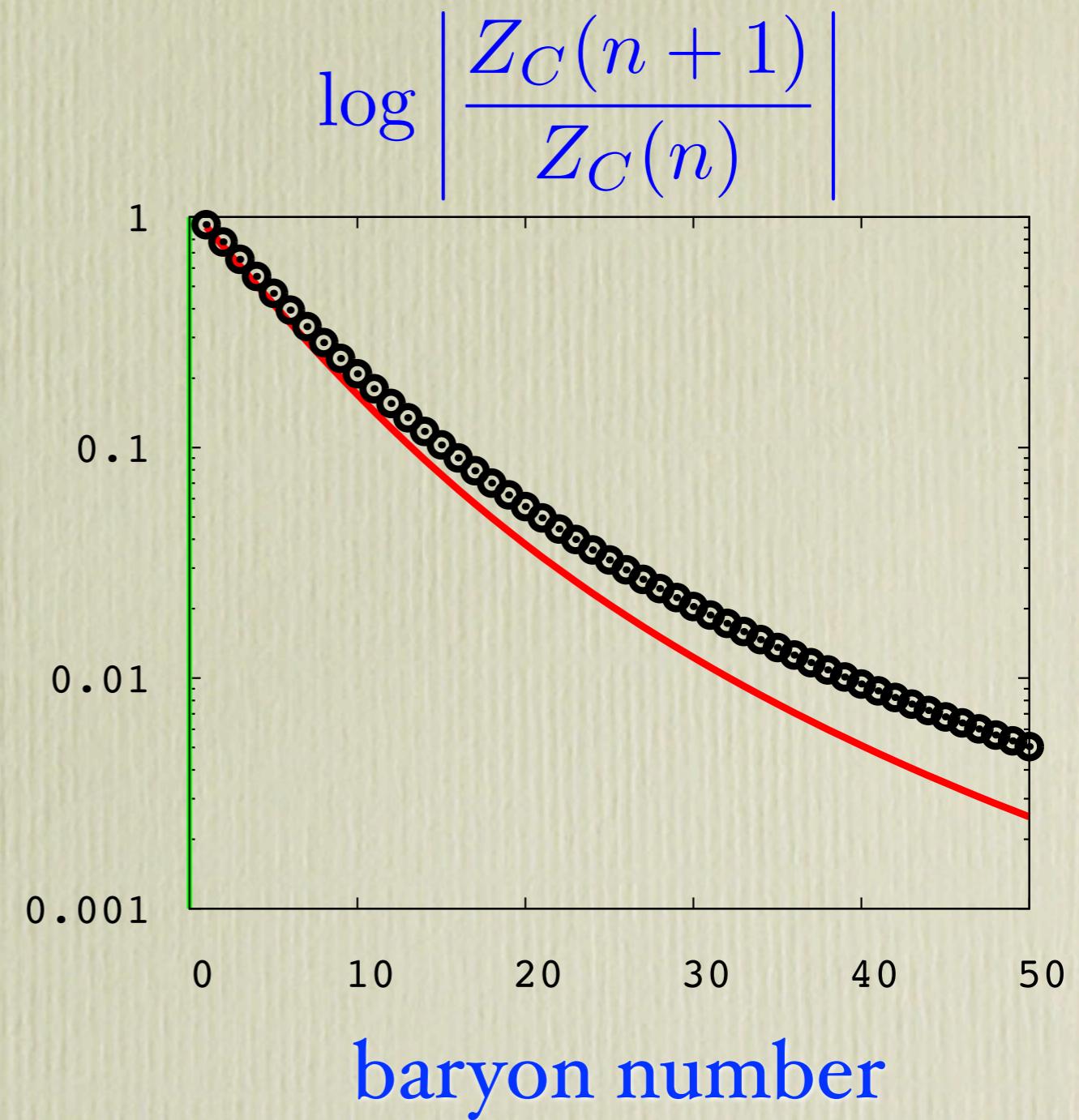
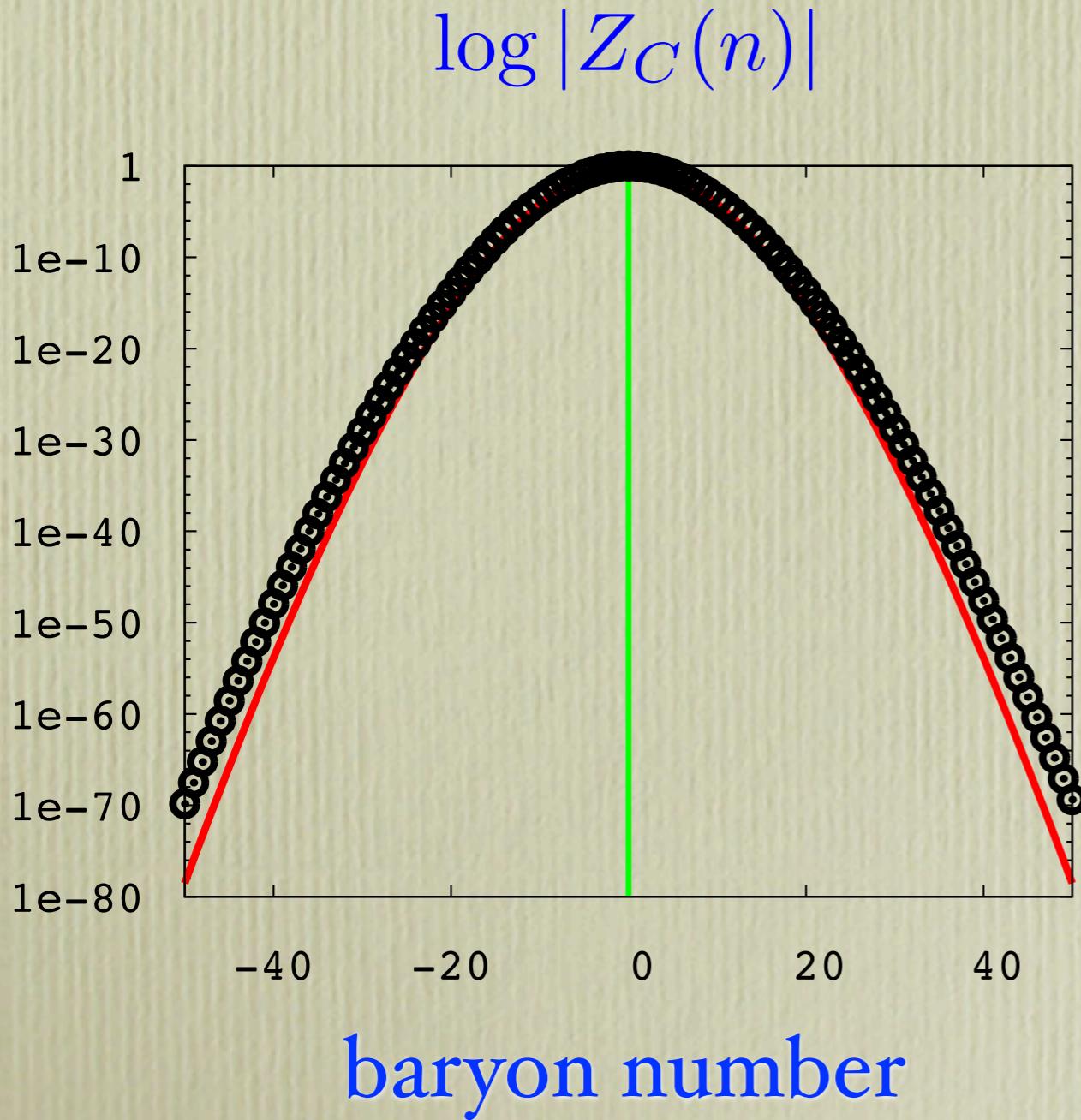
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Numerical results $Z_C(n)$

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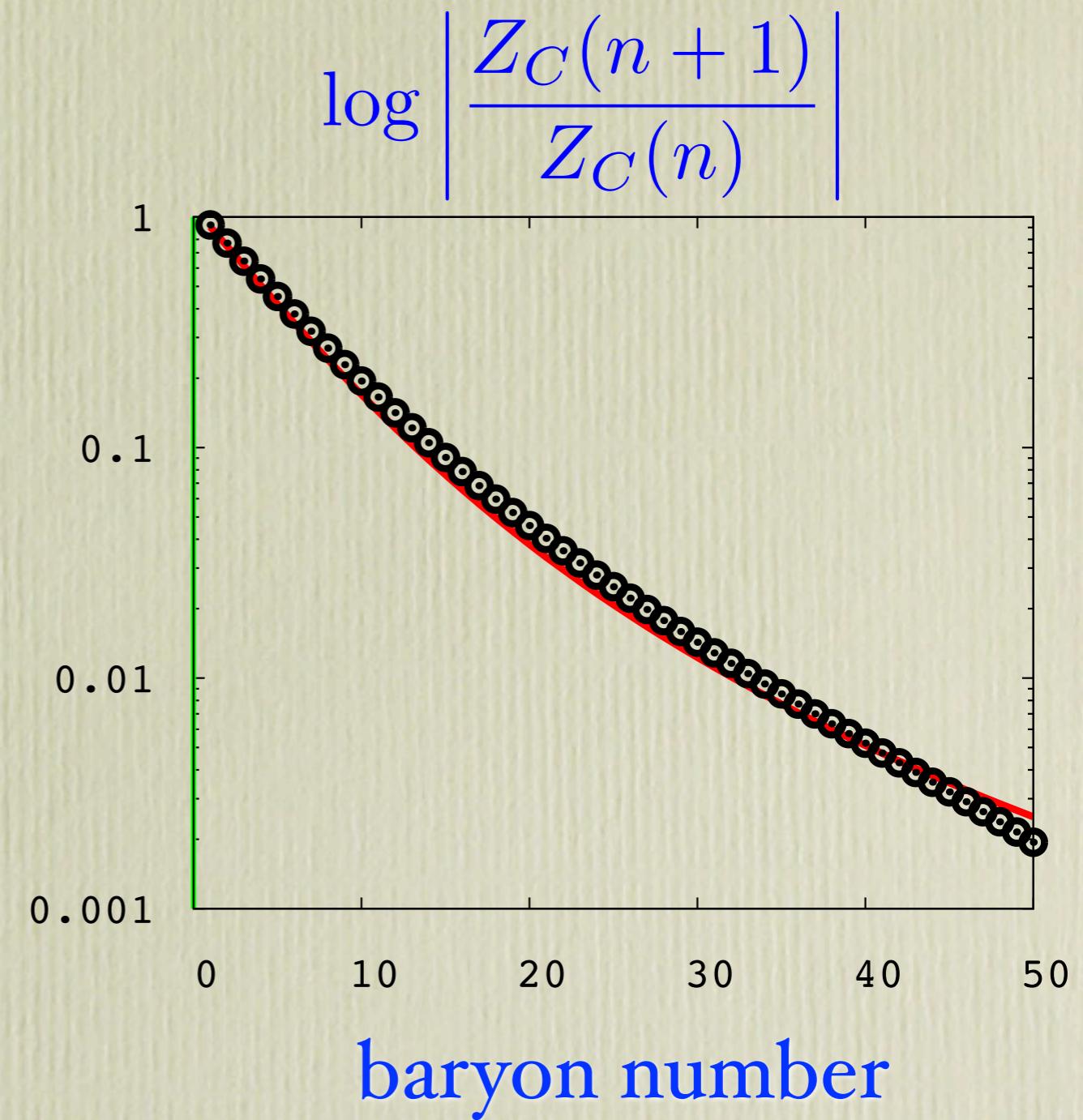
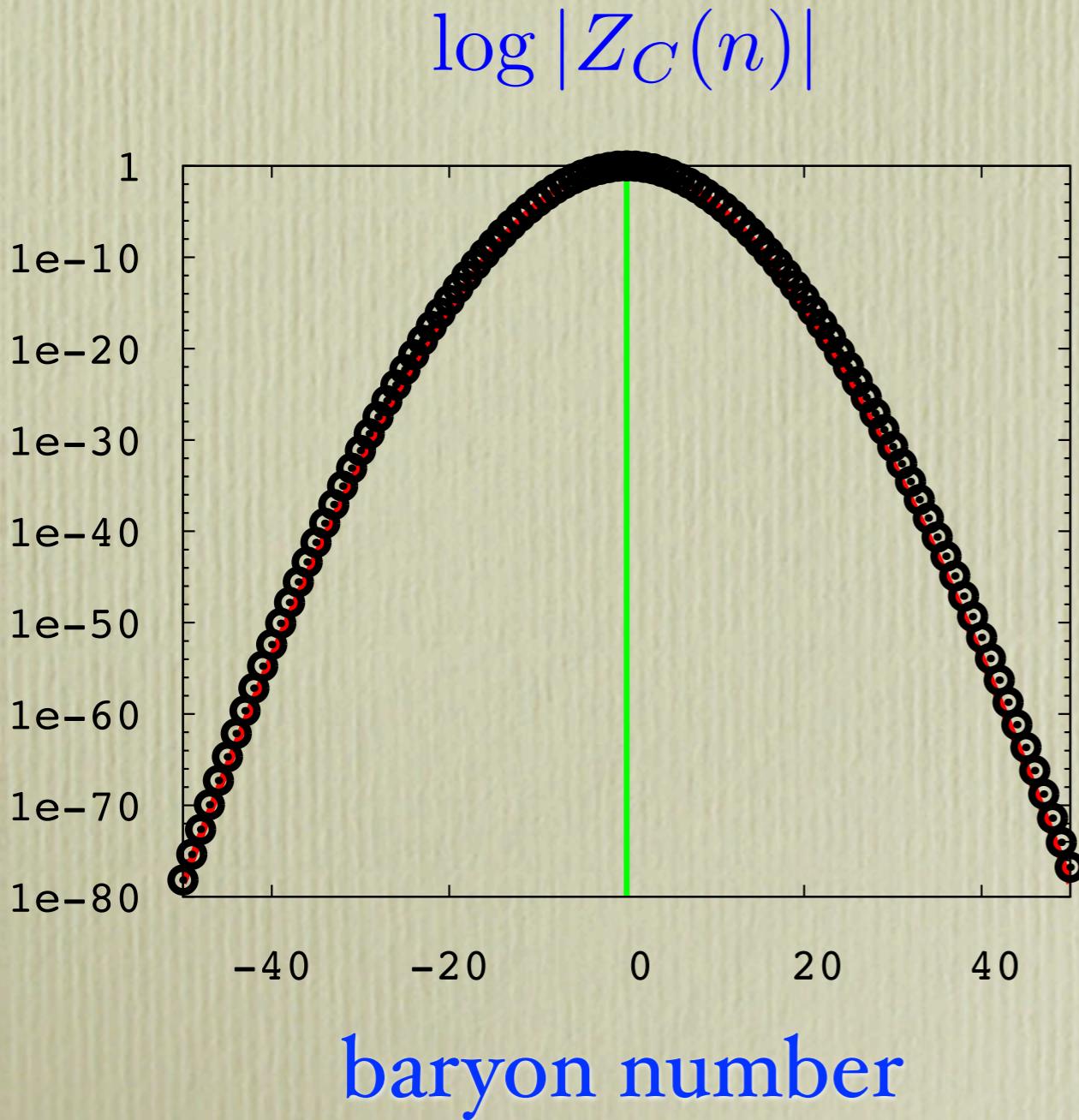
winding number=3 (number of hop =12)



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

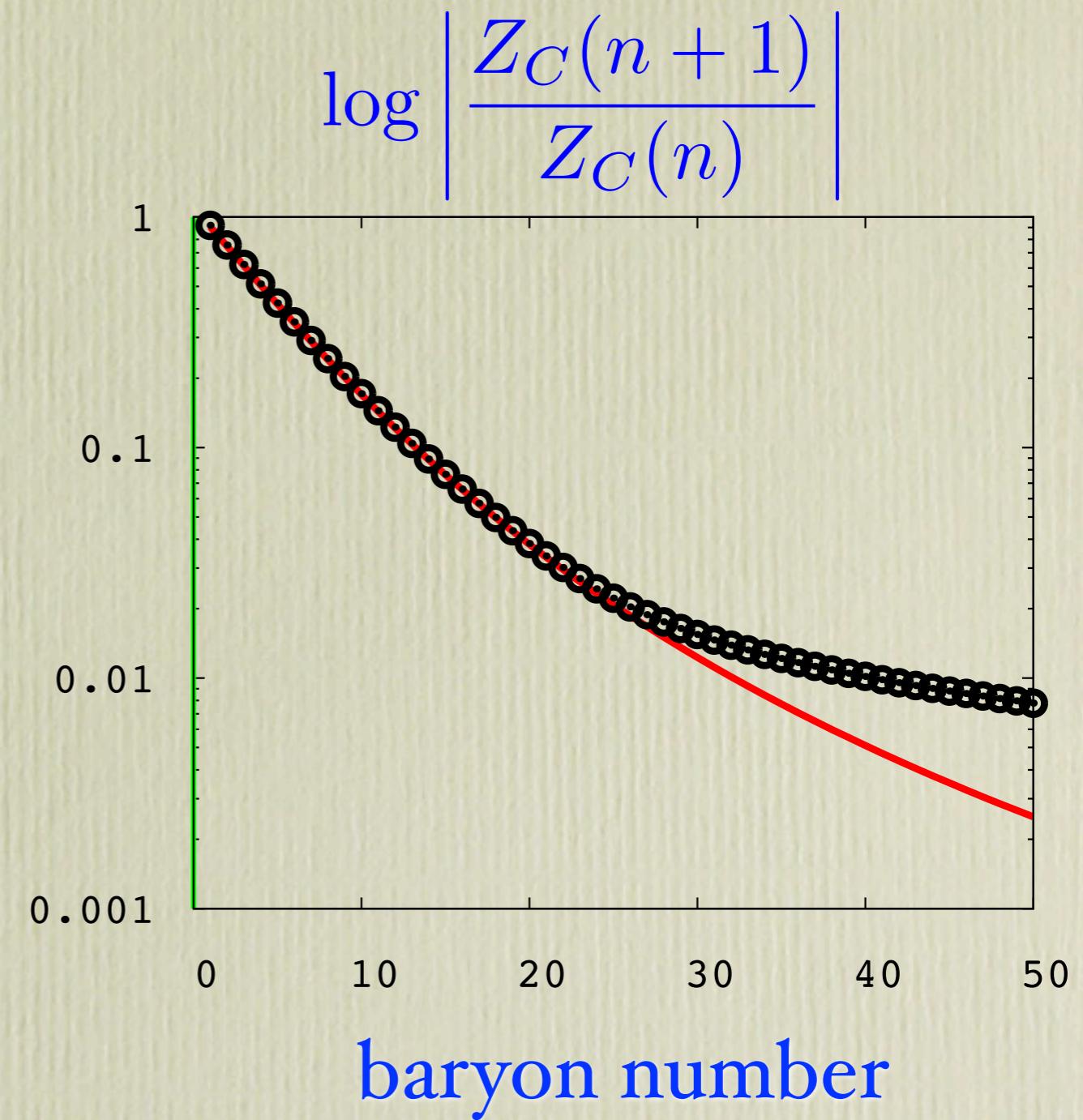
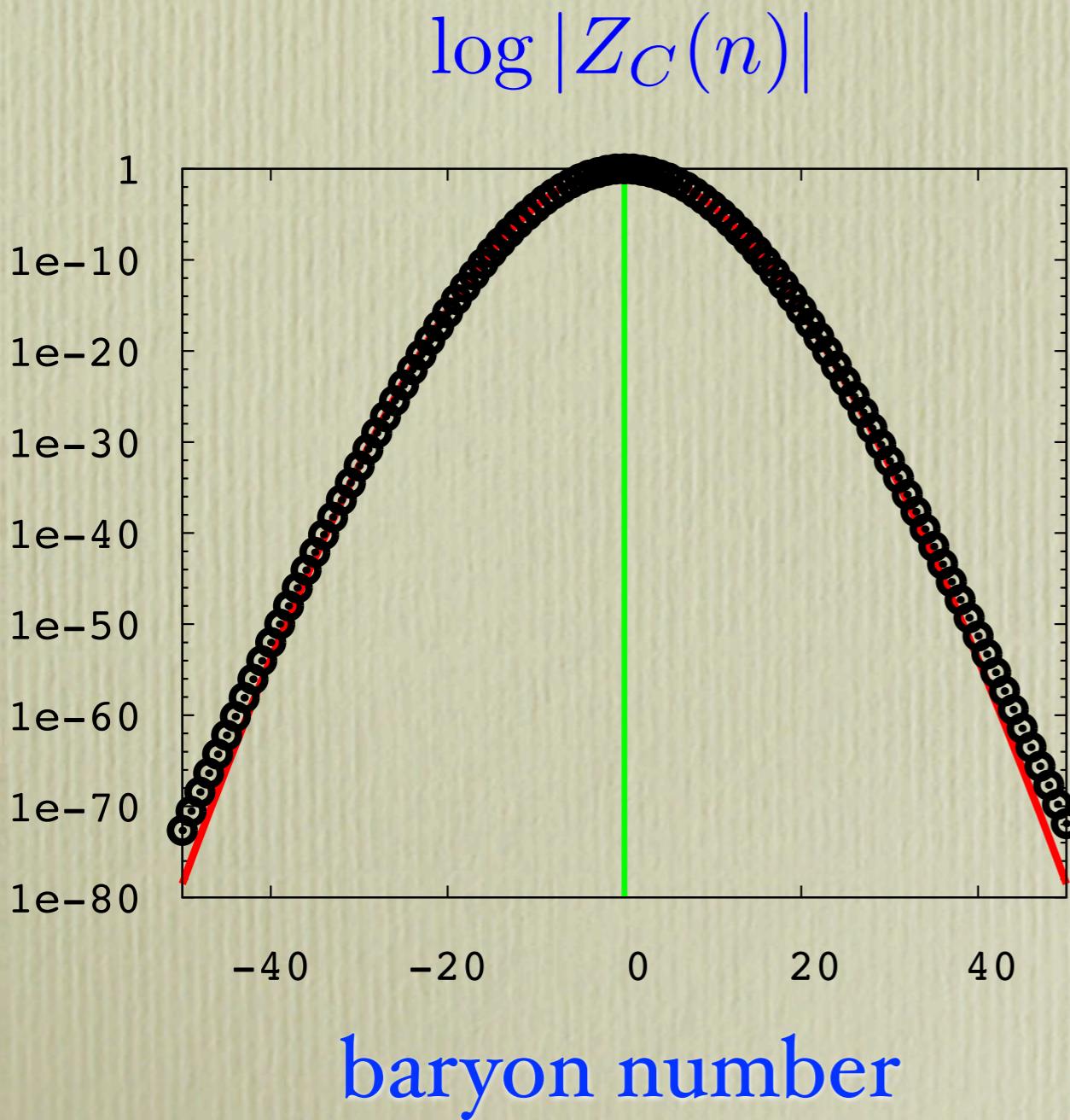
winding number=6 (number of hop =24)



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

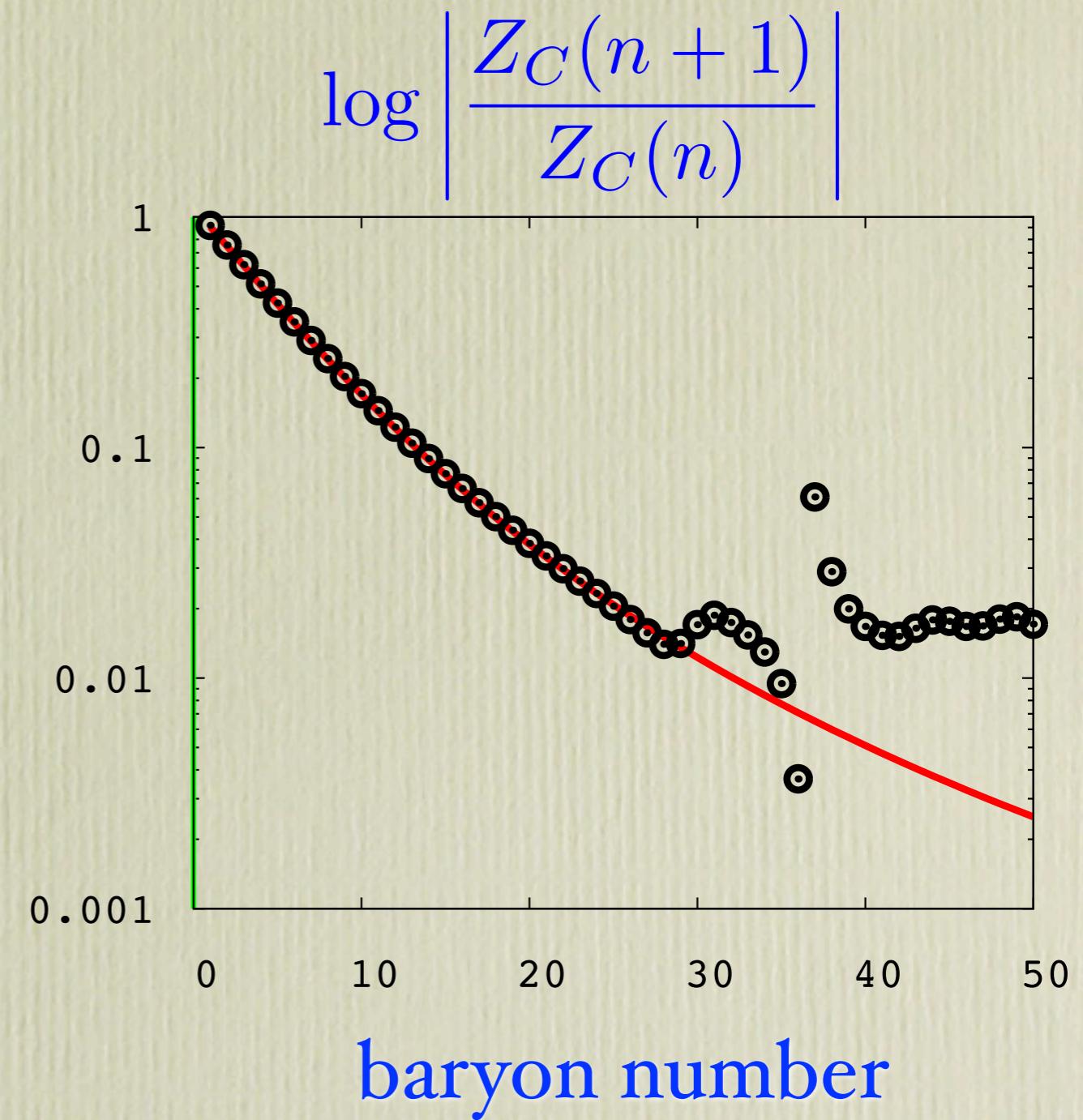
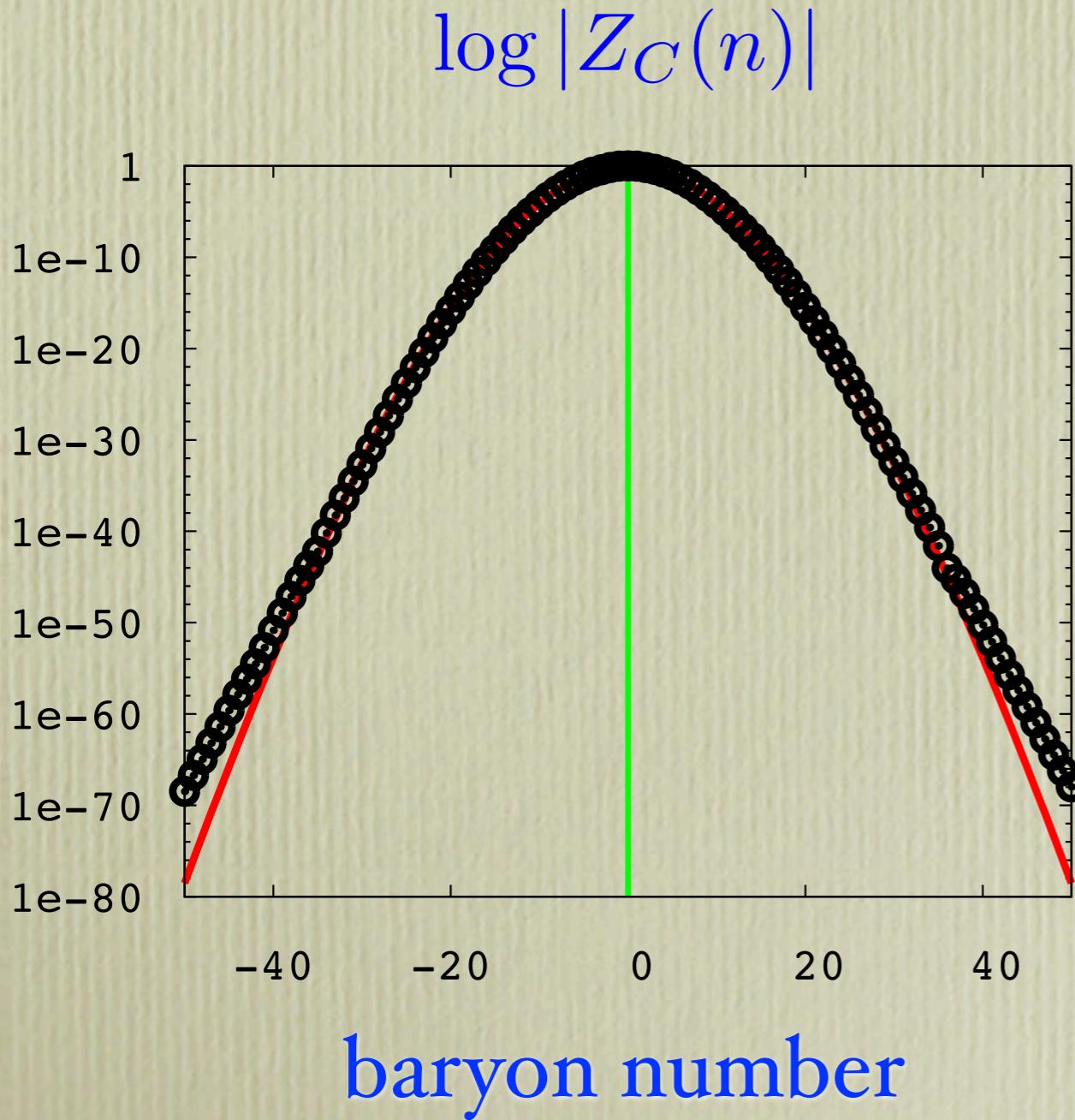
winding number=30 (number of hop = 120)



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

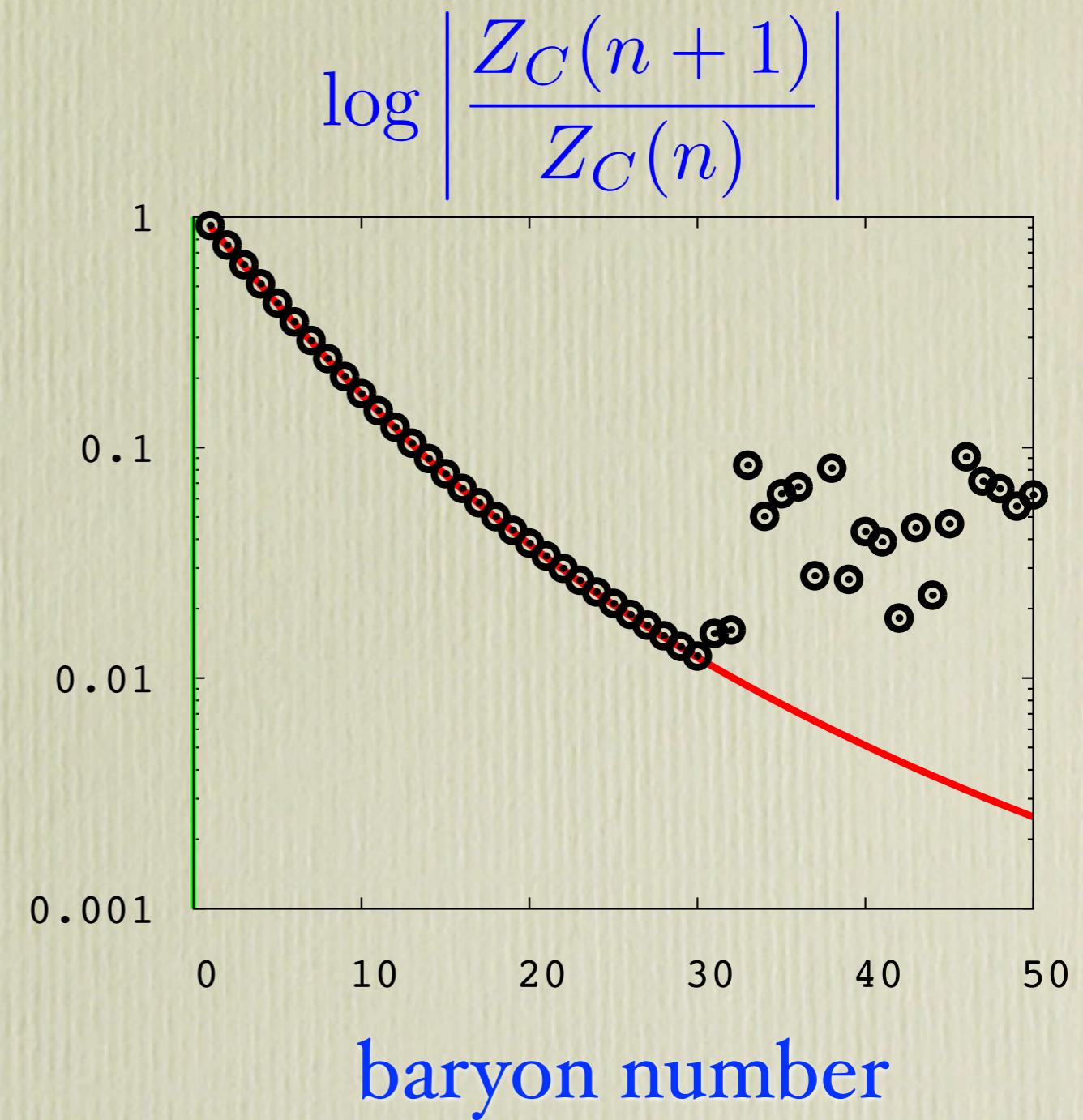
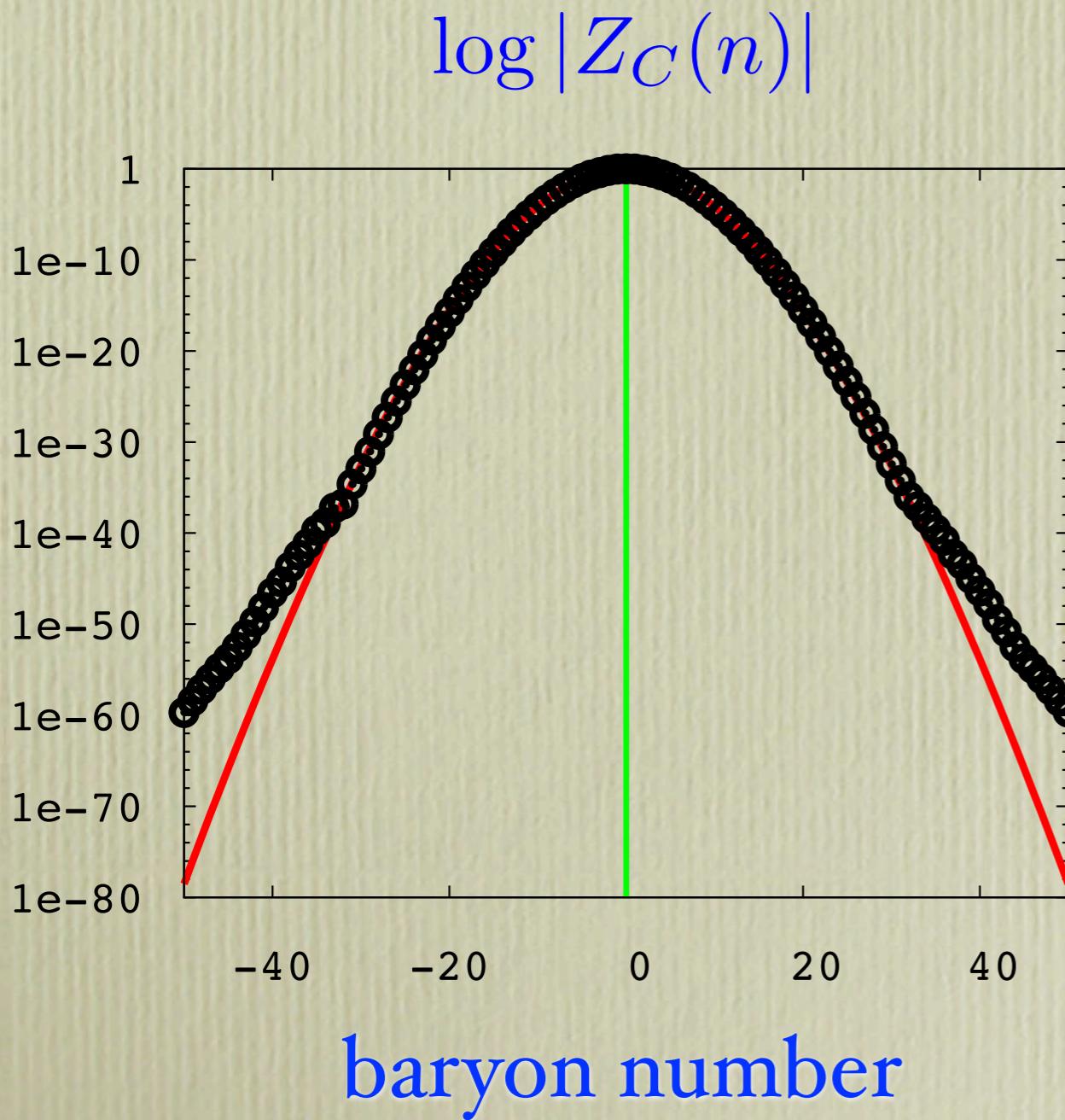
winding number=60 (number of hop =240)



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

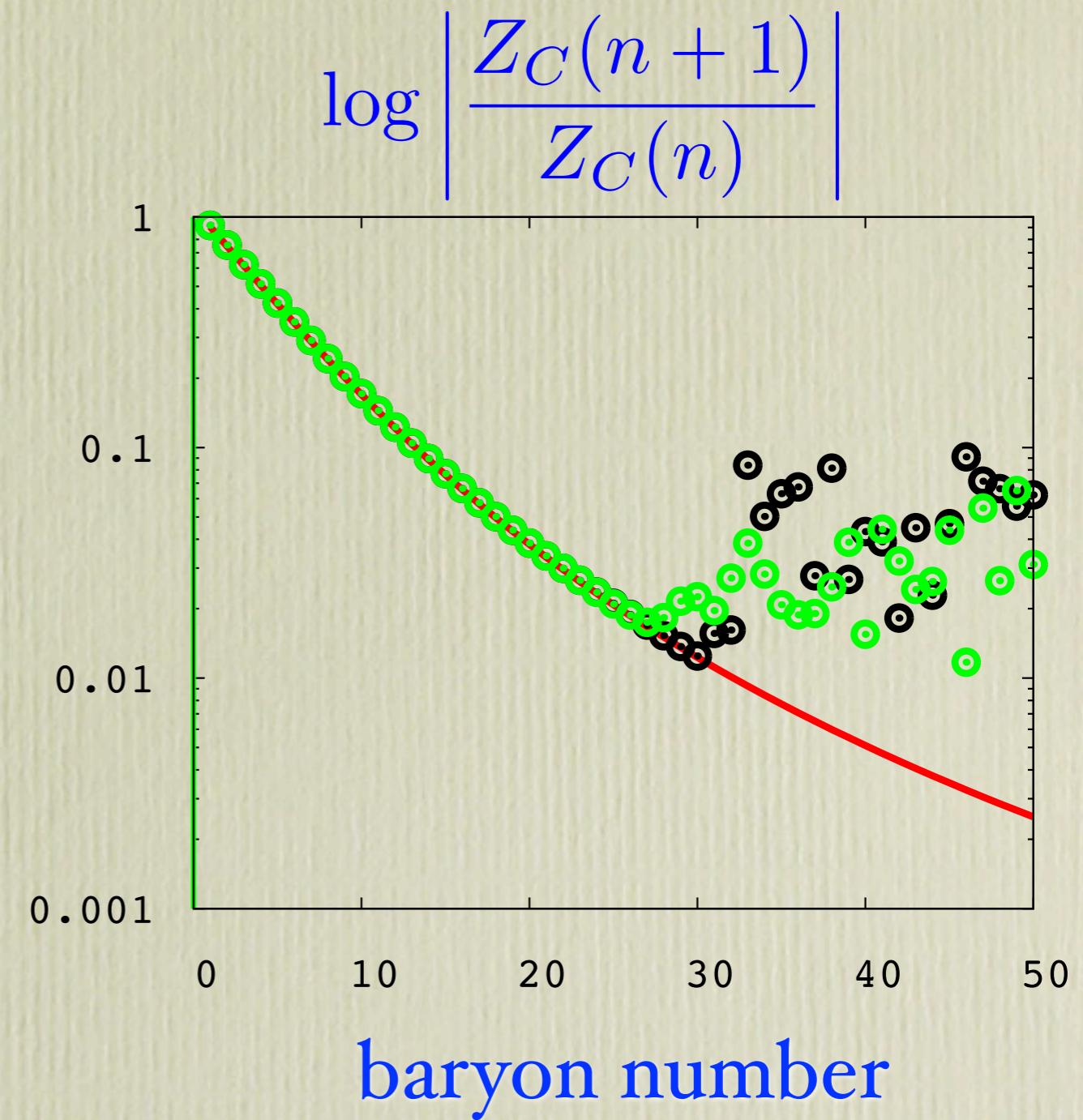
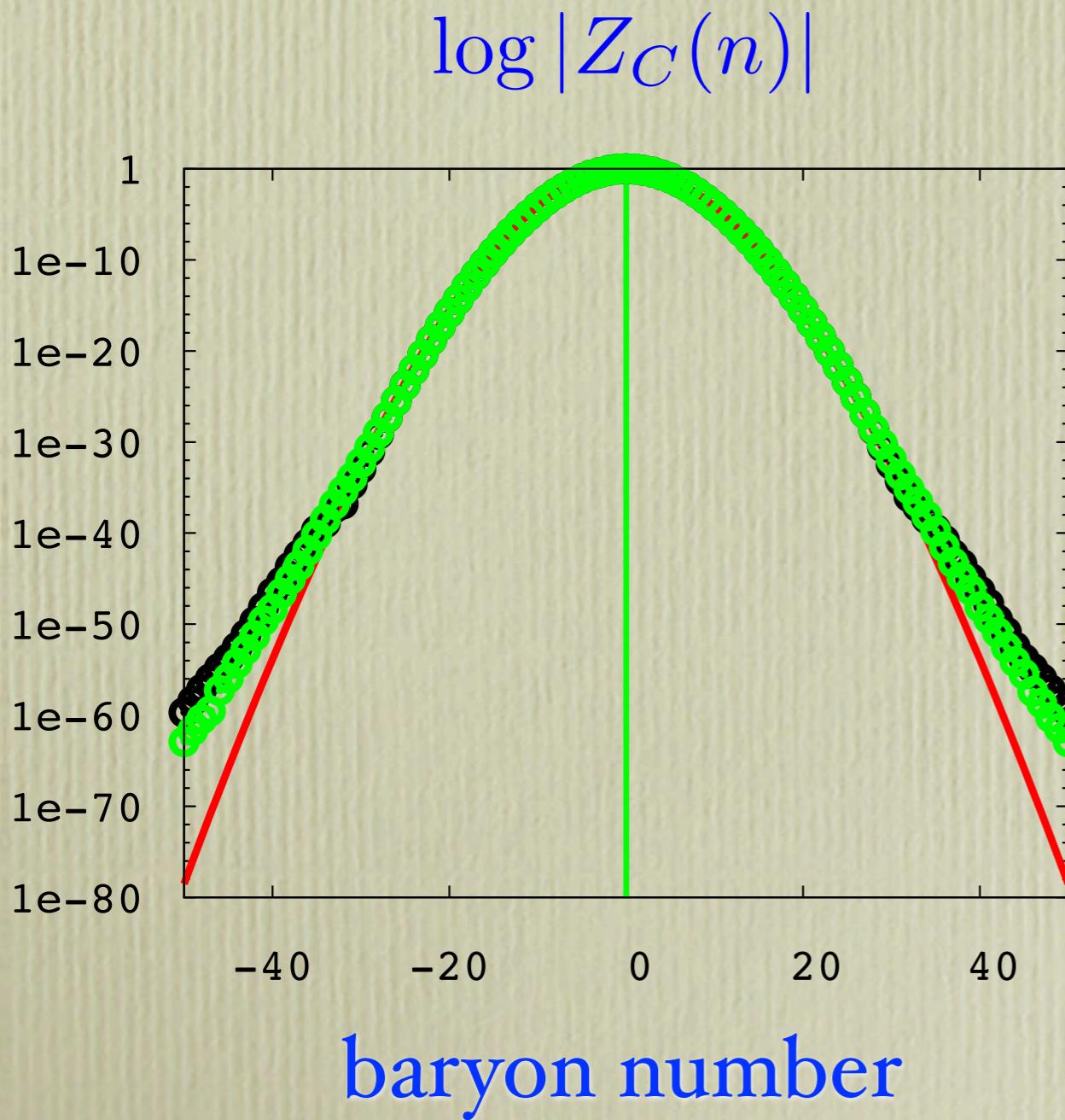
winding number=240 (number of hop =960)



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

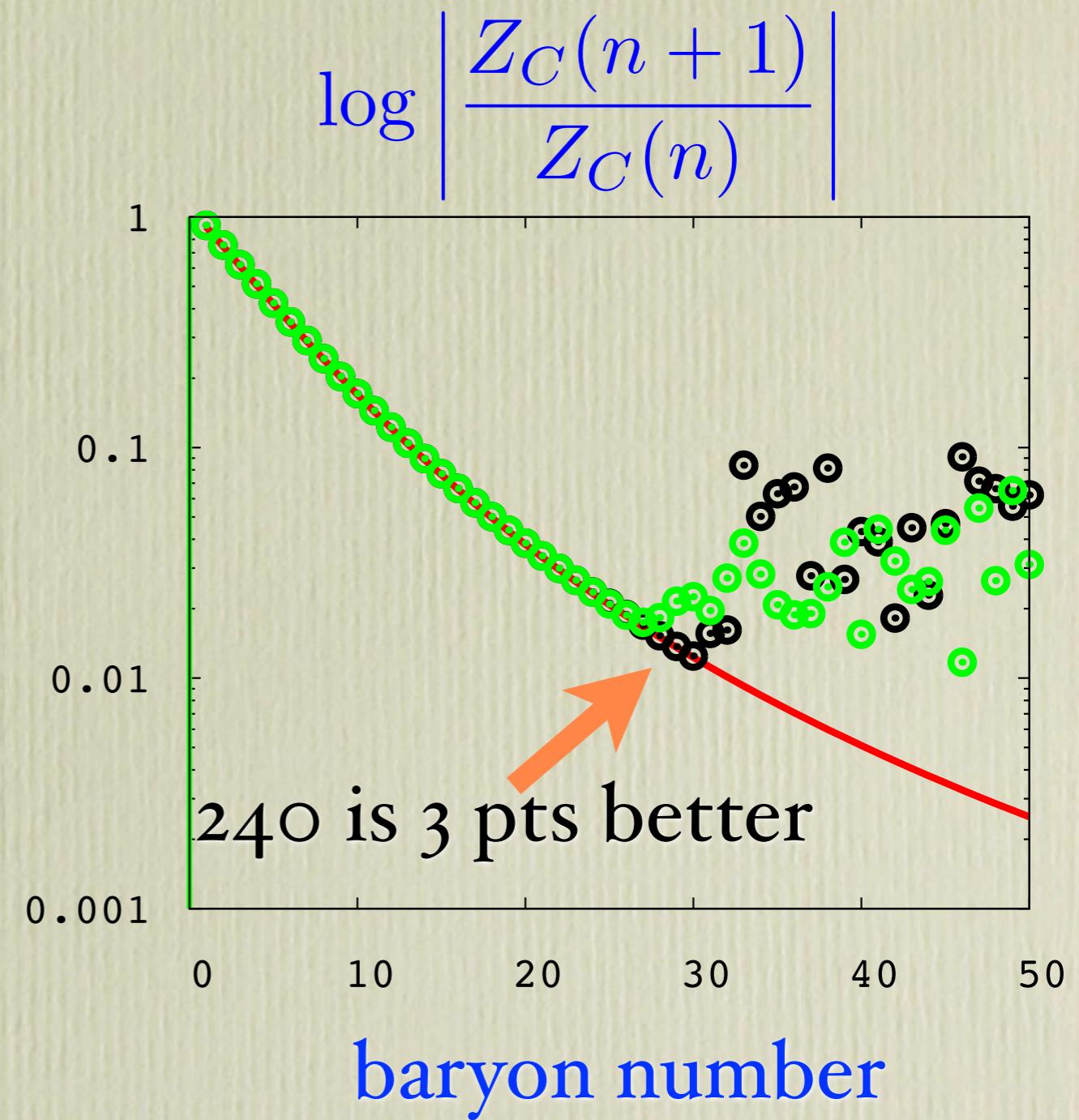
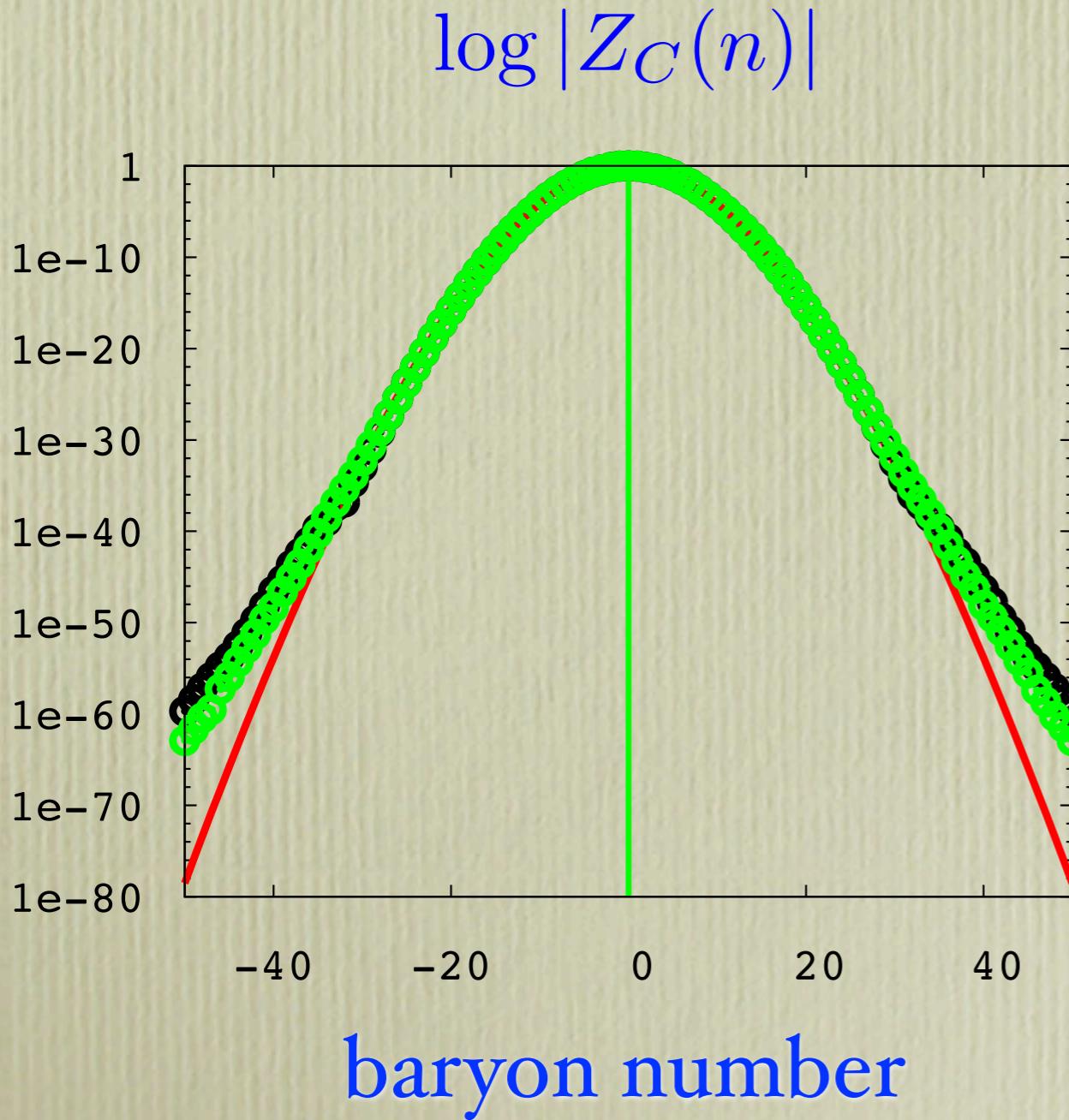
winding number=120 vs 240



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

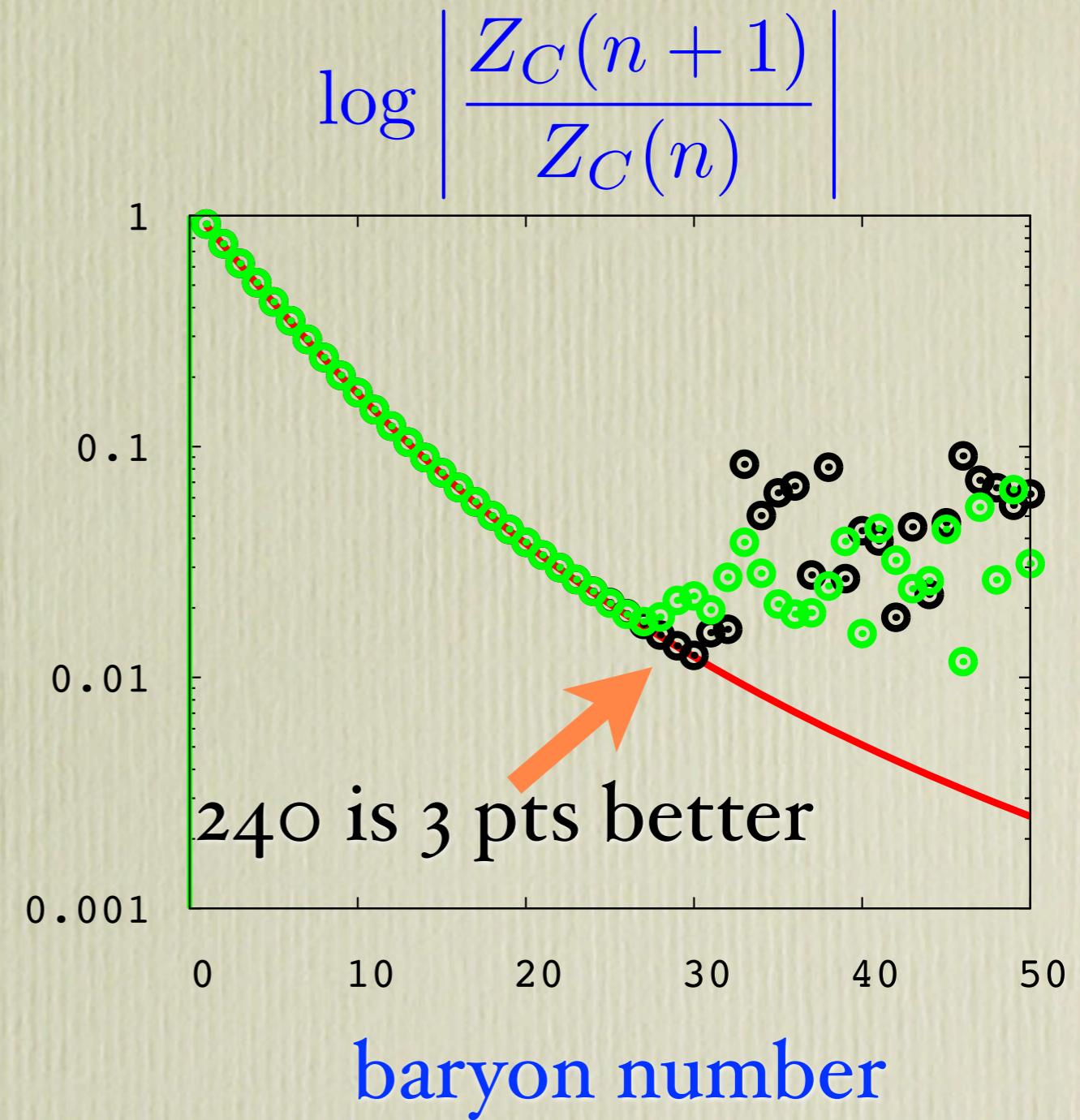
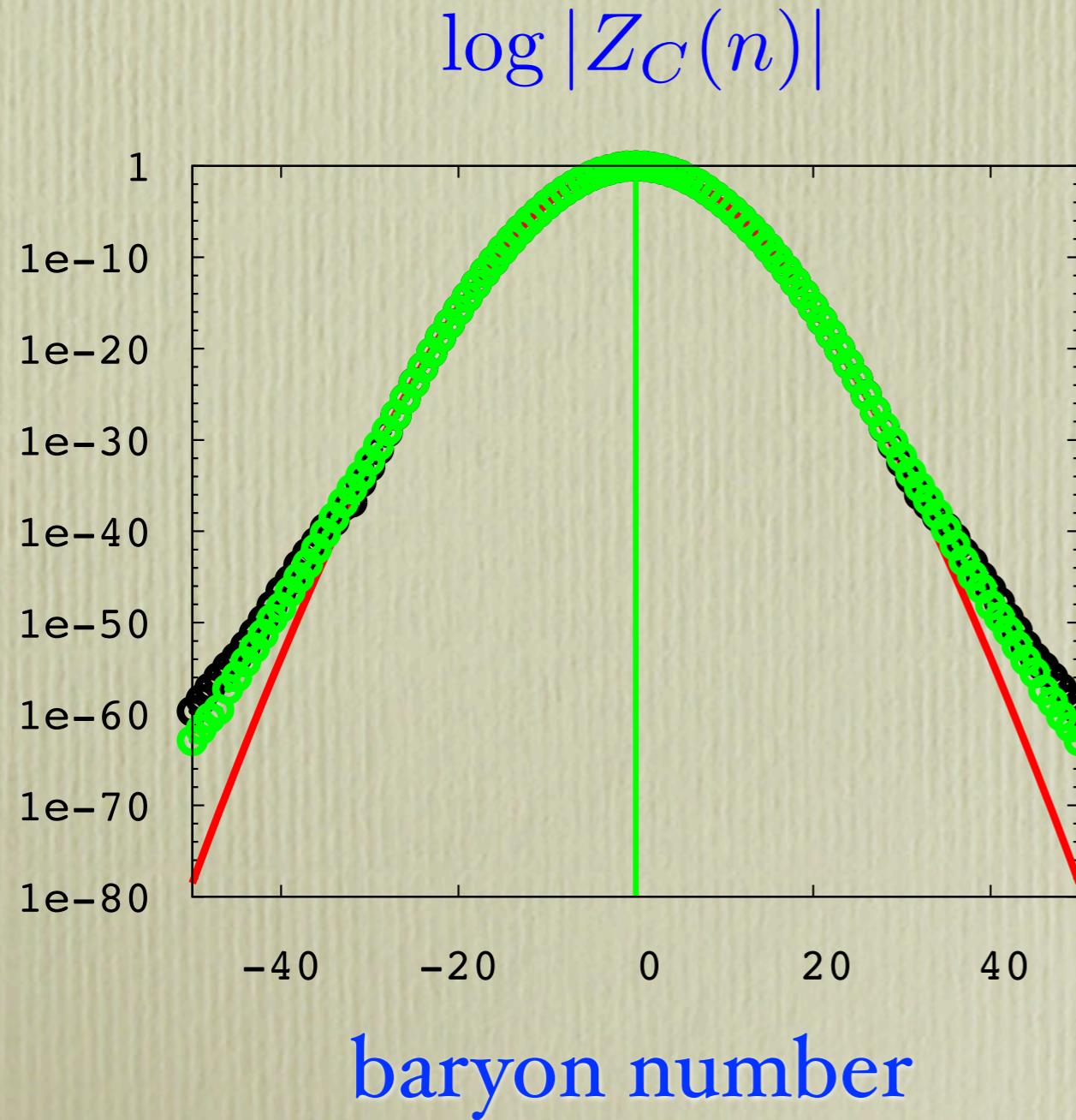
winding number=120 vs 240



Numerical results $Z_C(n)$

Multi precision for Fourier transformation

We adopt winding number=120



Numerical results $|Z_C(n)|$

Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

Numerical results $|Z_C(n)|$

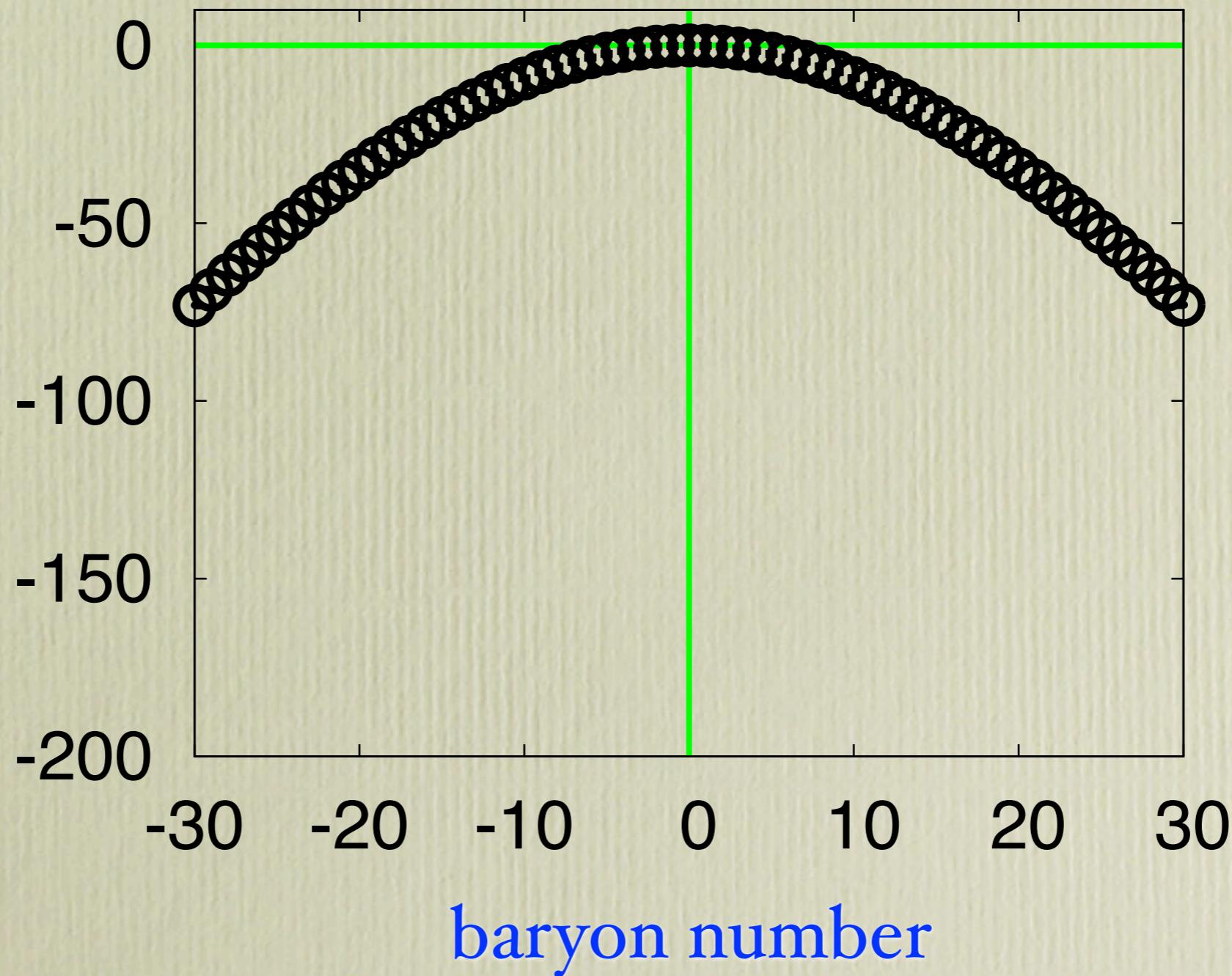
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 2.1$$

$$\mu = 0$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

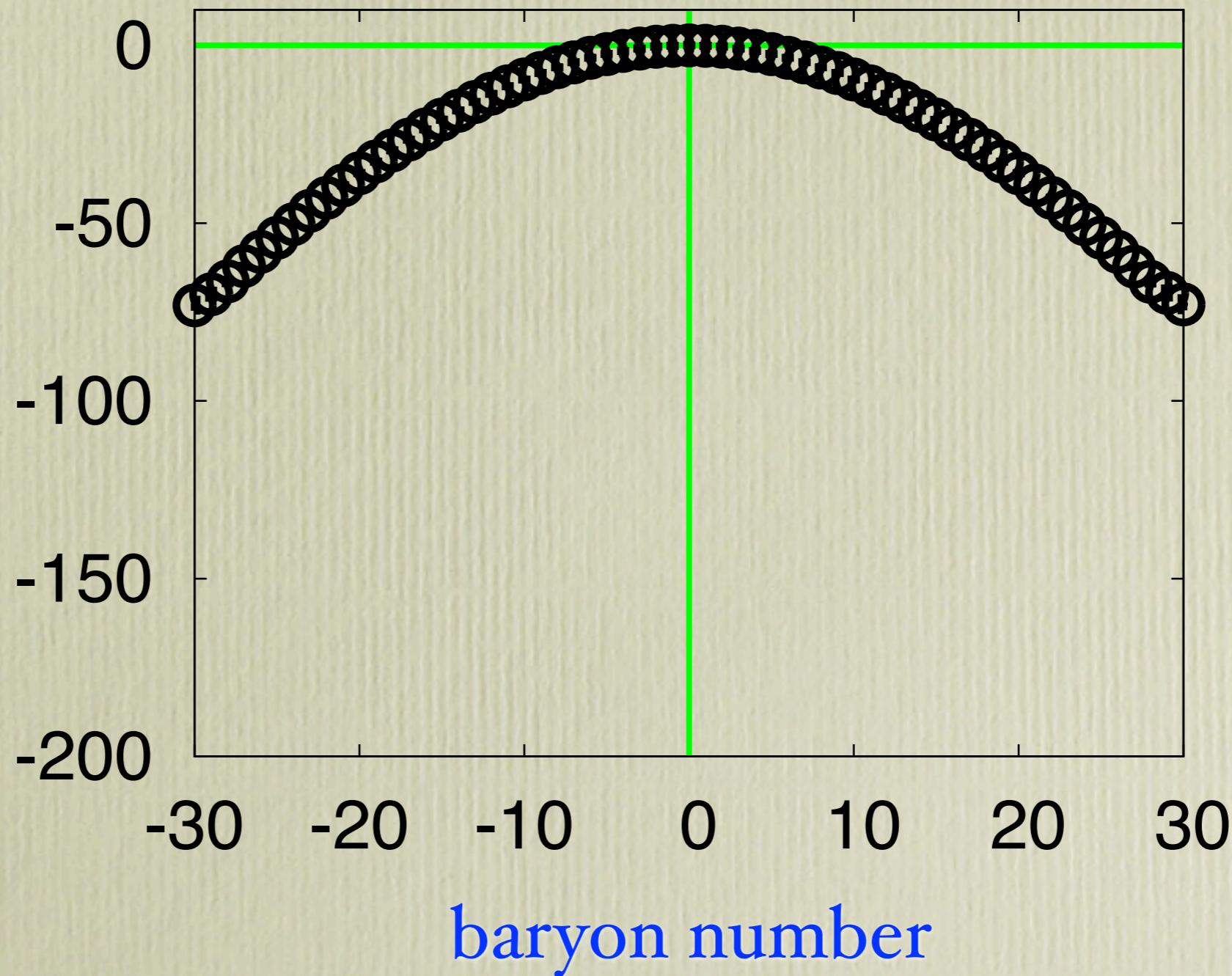
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.9$$

$$\mu = 0$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

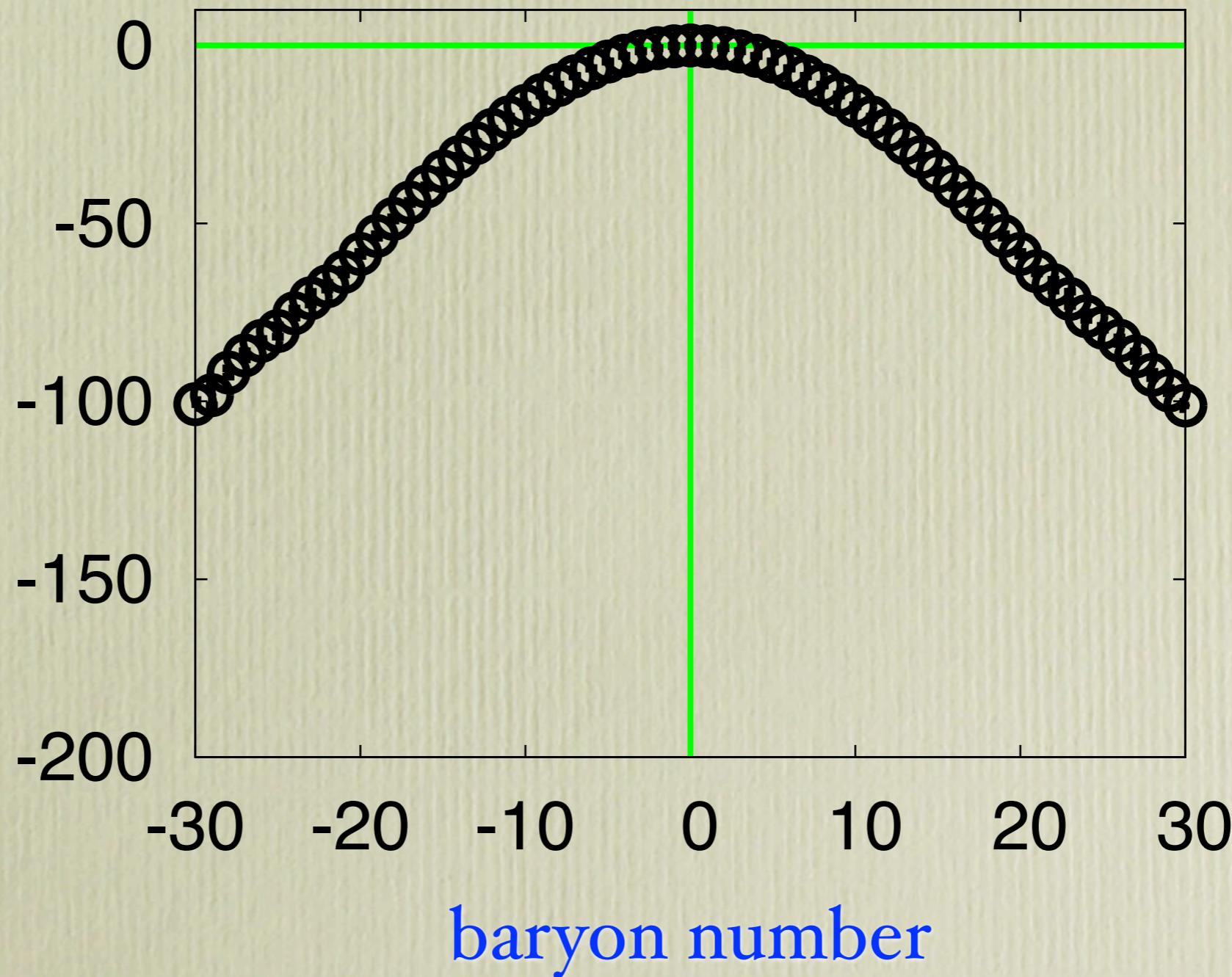
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.7$$

$$\mu = 0$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

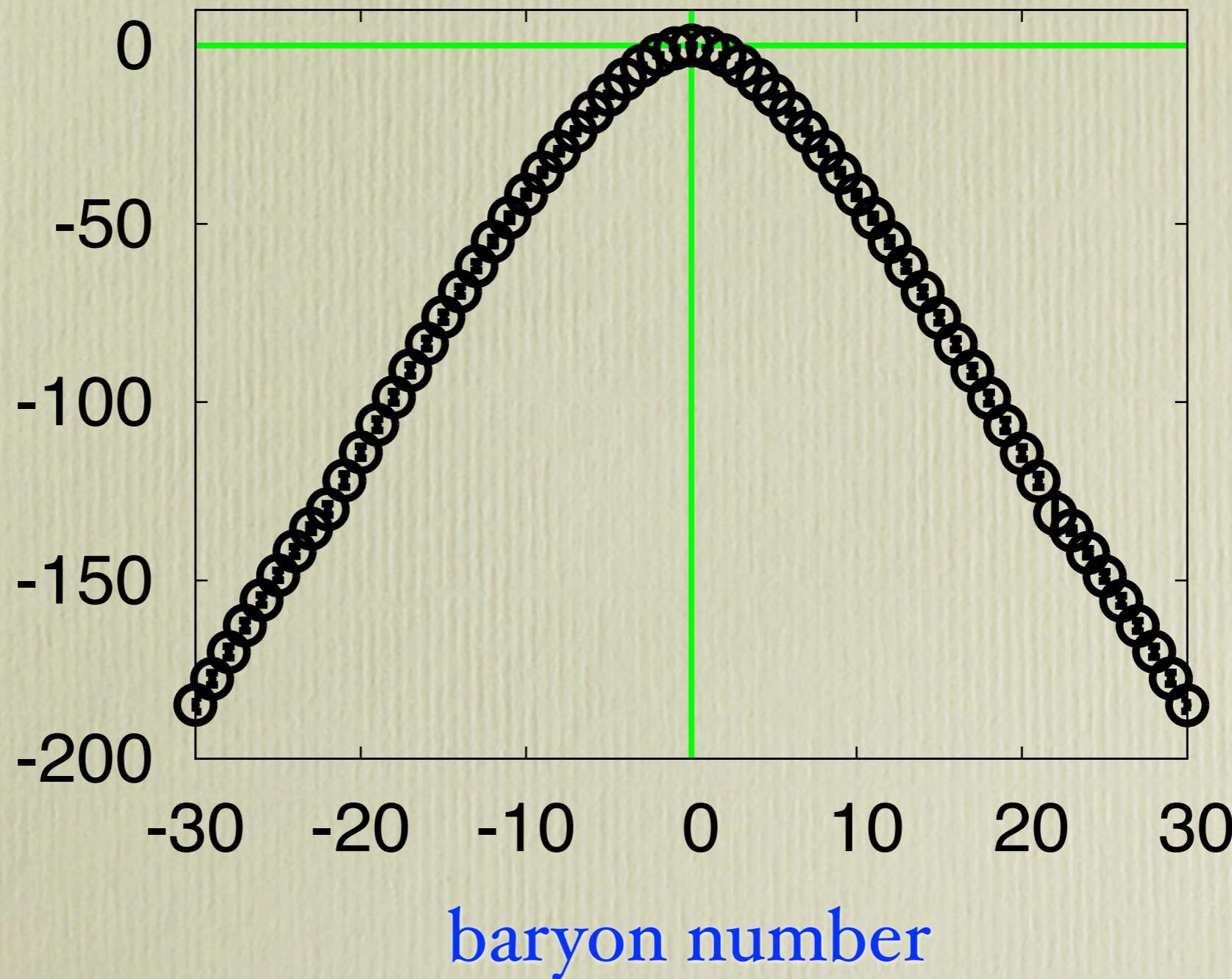
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$\beta = 1.5$

$\mu = 0$

$\log |Z_C(\beta, n)|$



Numerical results $|Z_C(n)|$

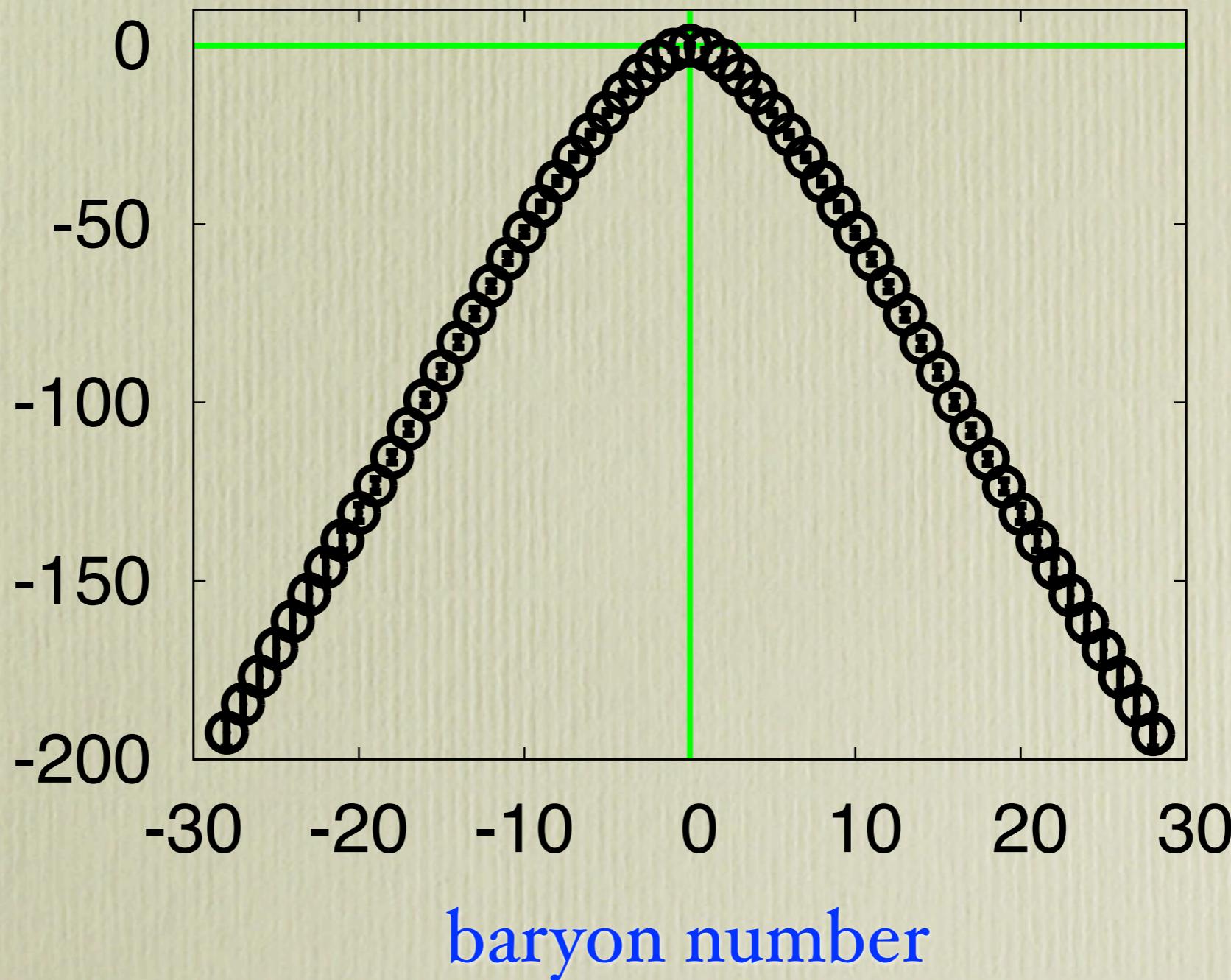
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.3$$

$$\mu = 0$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

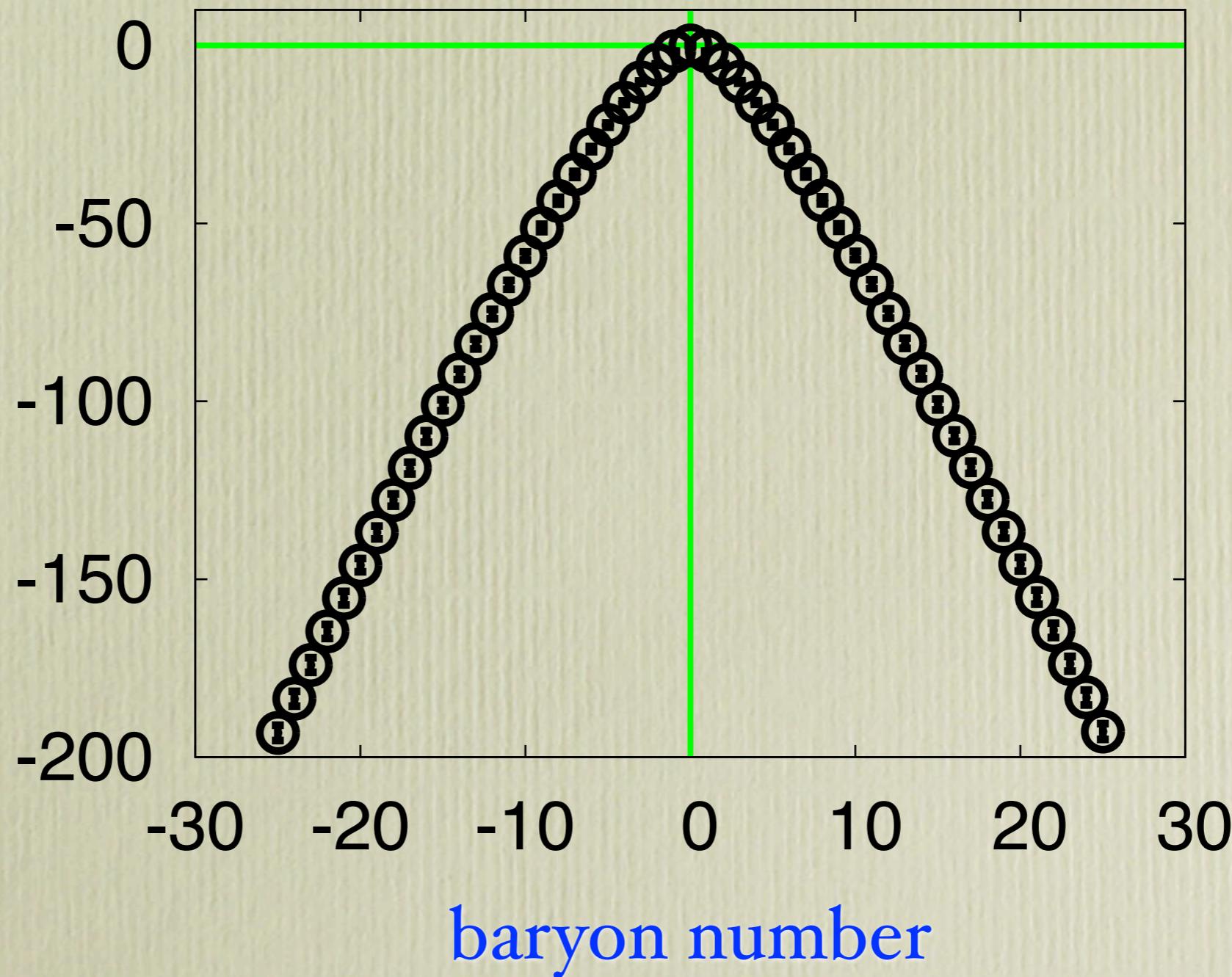
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.1$$

$$\mu = 0$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

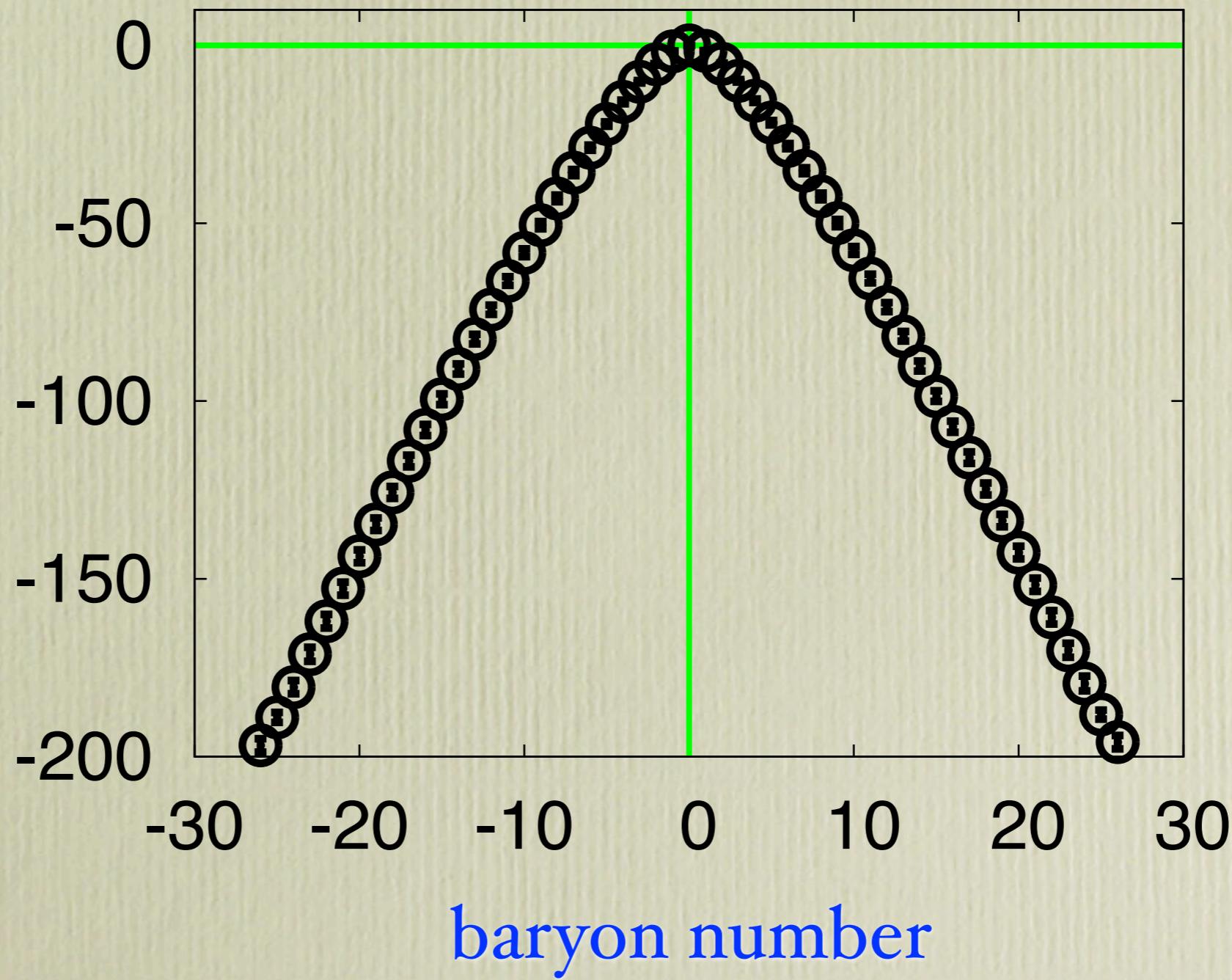
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 0.9$$

$$\mu = 0$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

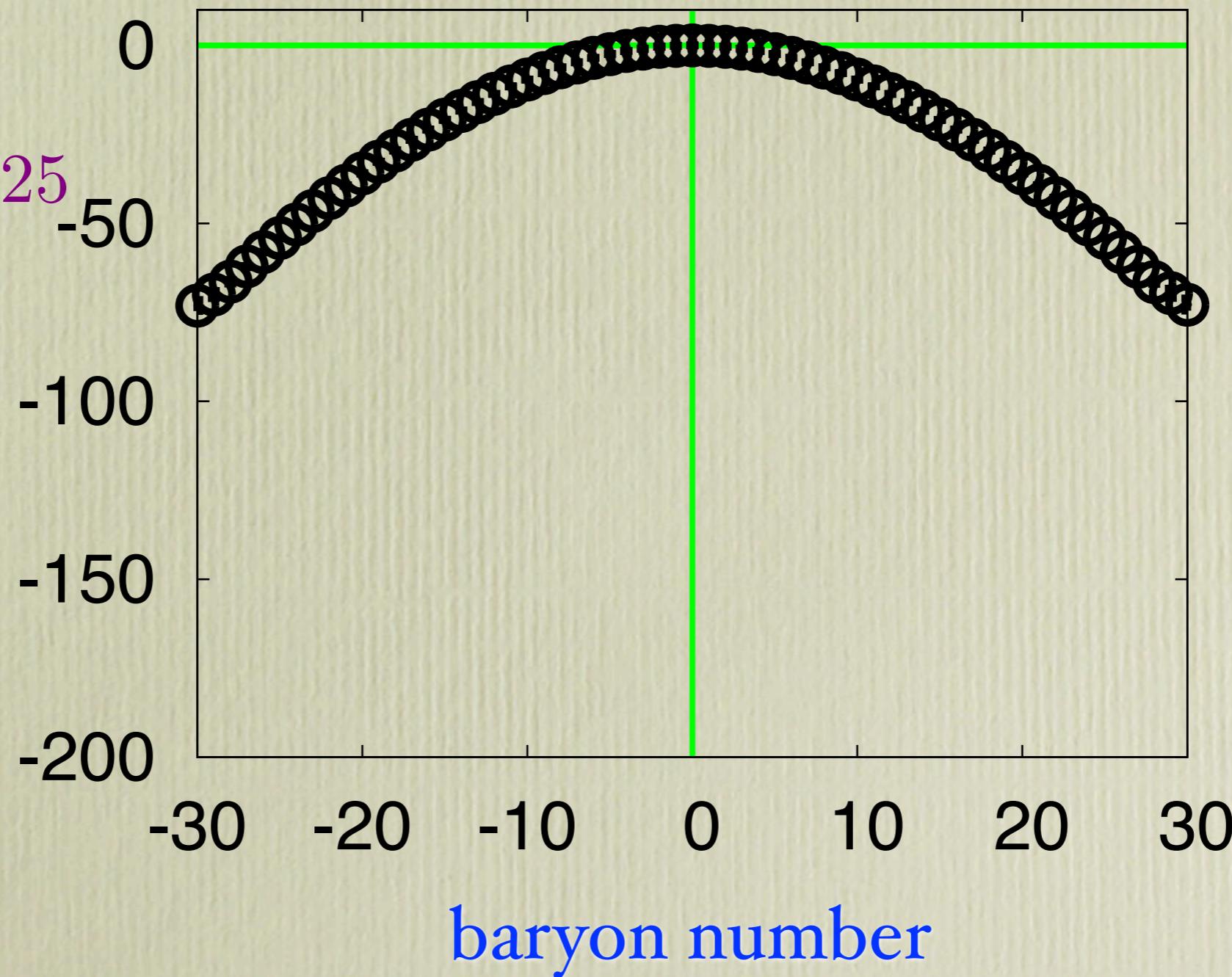
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.9$$

$$\mu_I = 0.125, 0.25$$

$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

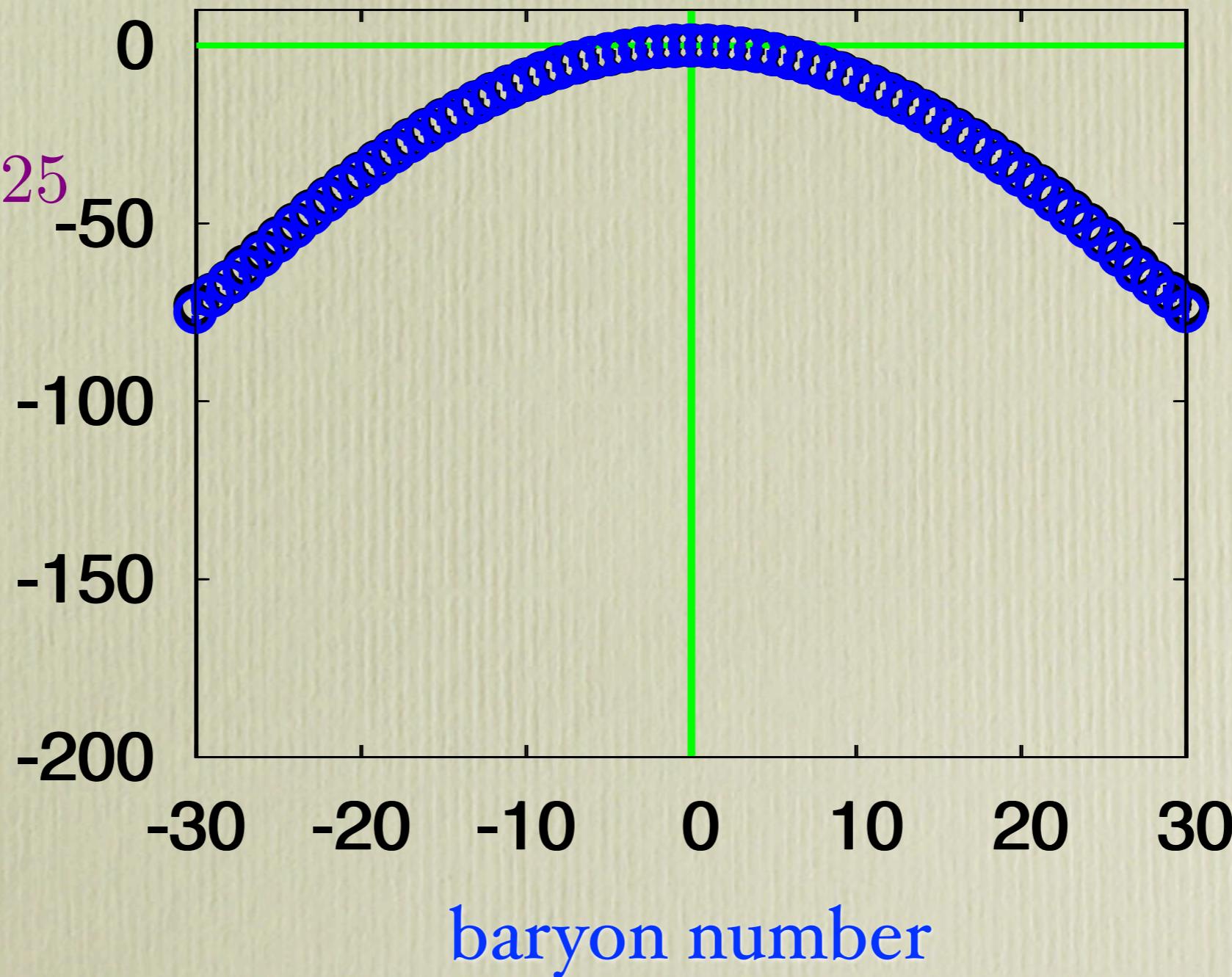
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Numerical results $|Z_C(n)|$

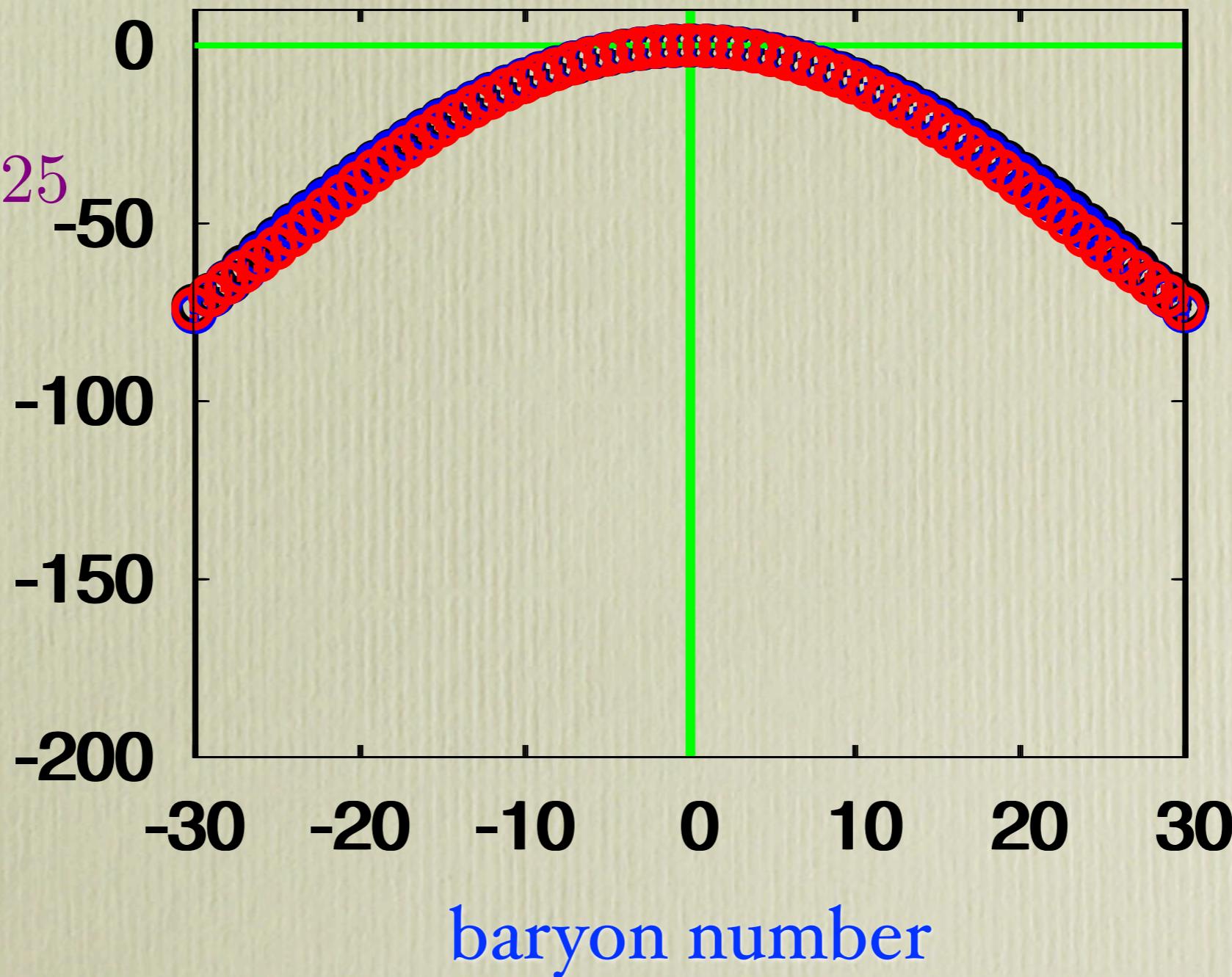
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$$\log |Z_C(\beta, n)|$$



Numerical results $|Z_C(n)|$

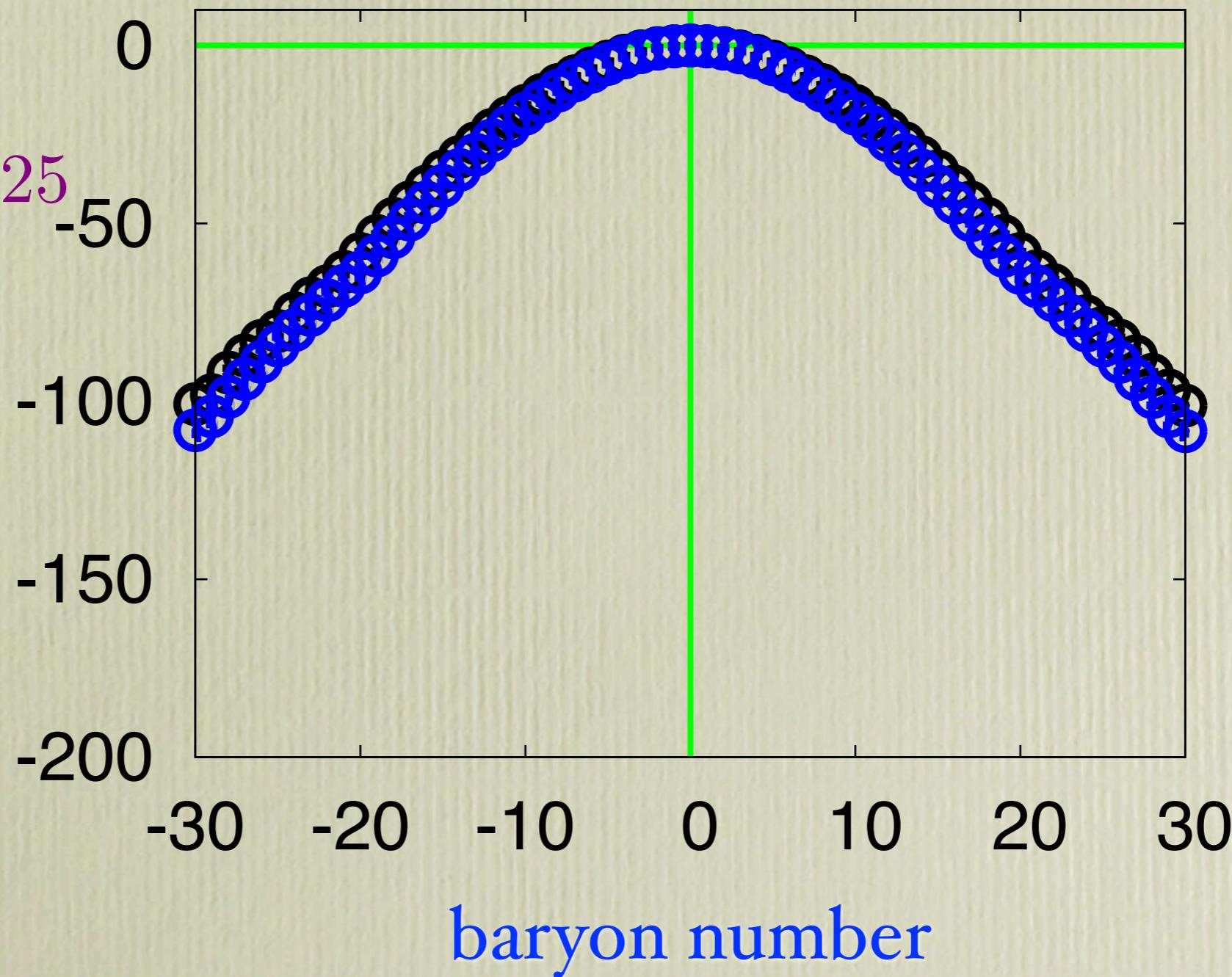
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Numerical results $|Z_C(n)|$

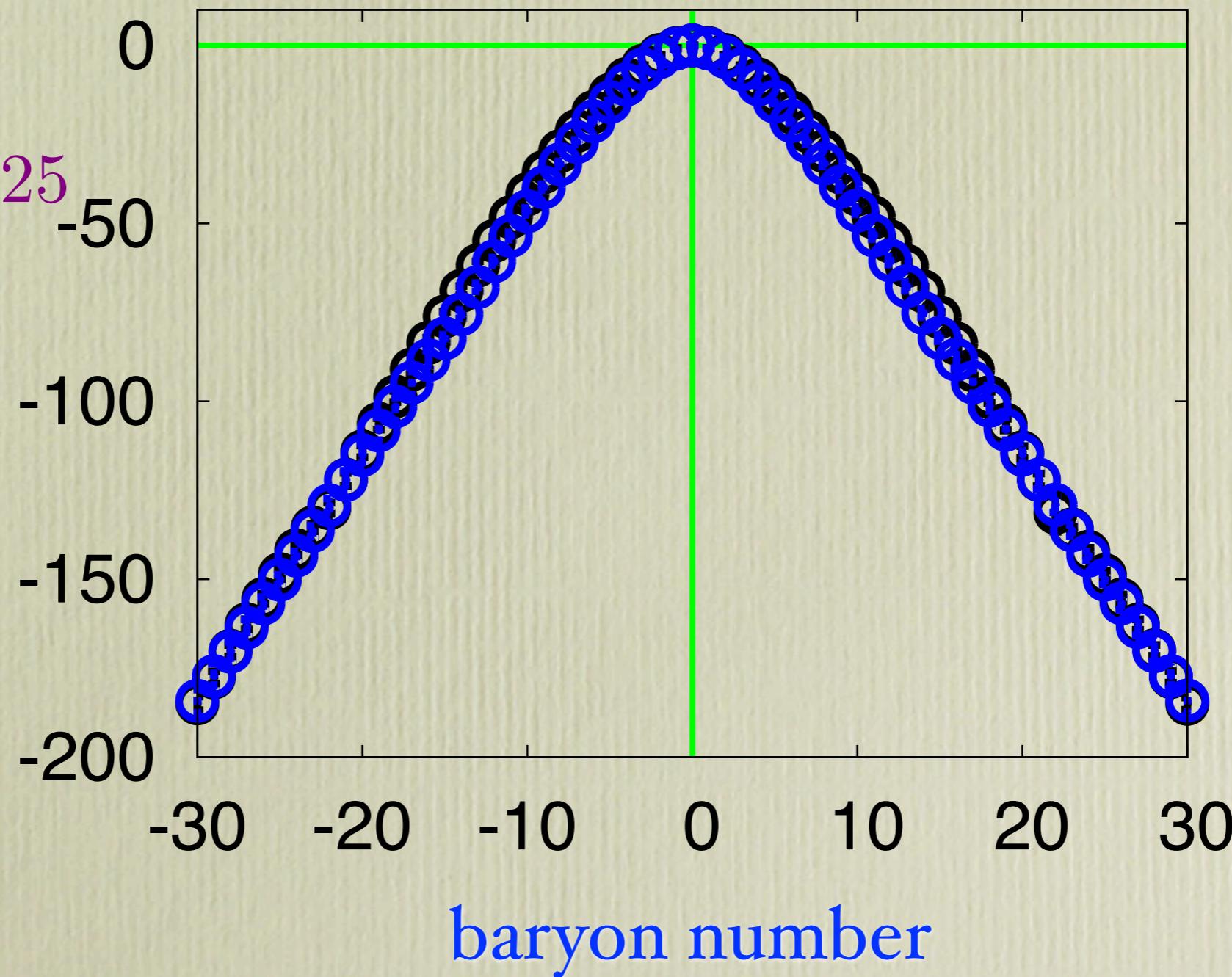
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.5$$

$$\mu_I = 0.125, 0.25$$

$$\log |Z_C(\beta, n)|$$



Numerical results Phase(Zc(n))

Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

Numerical results Phase($Z_C(n)$)

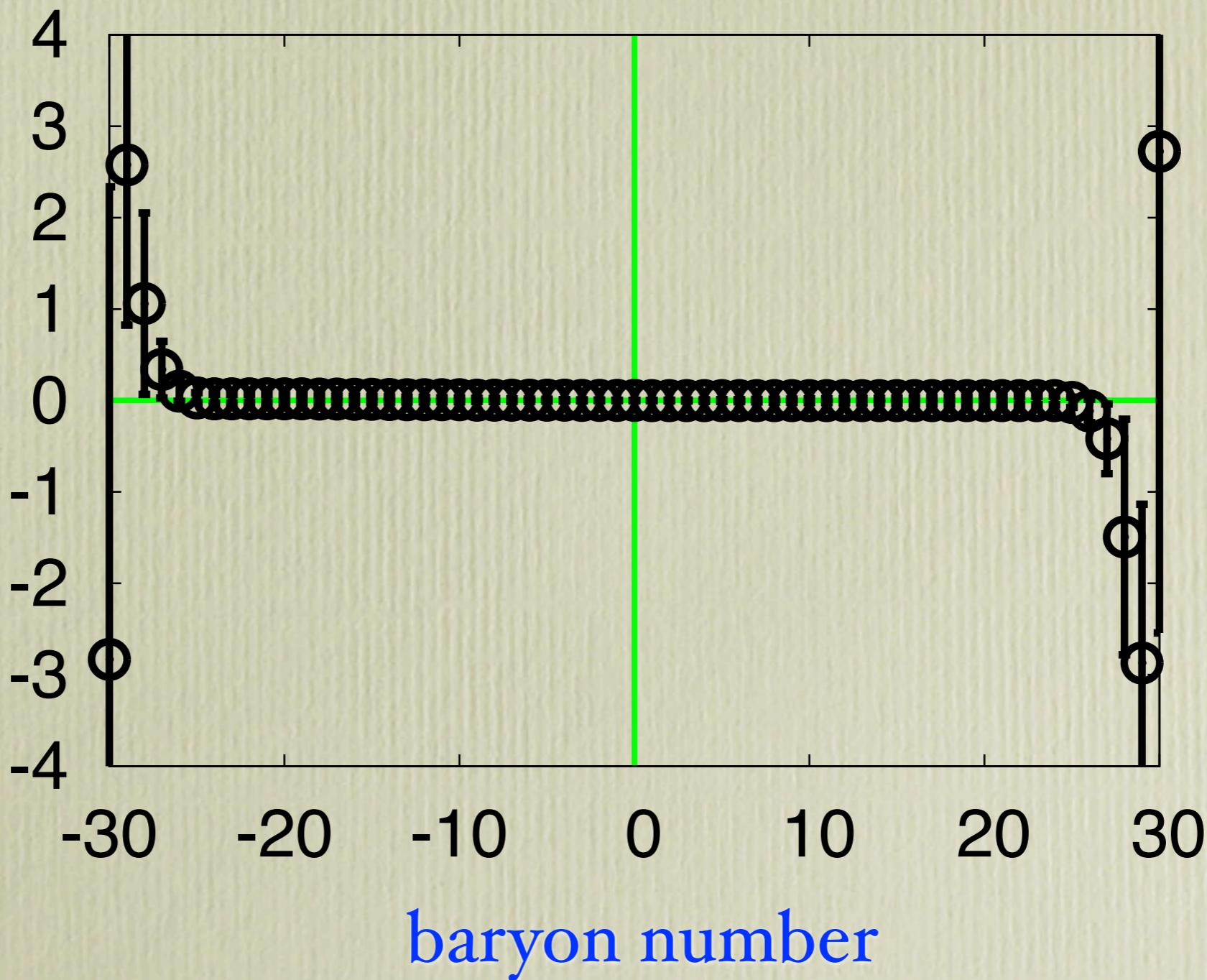
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$\mu = 0$

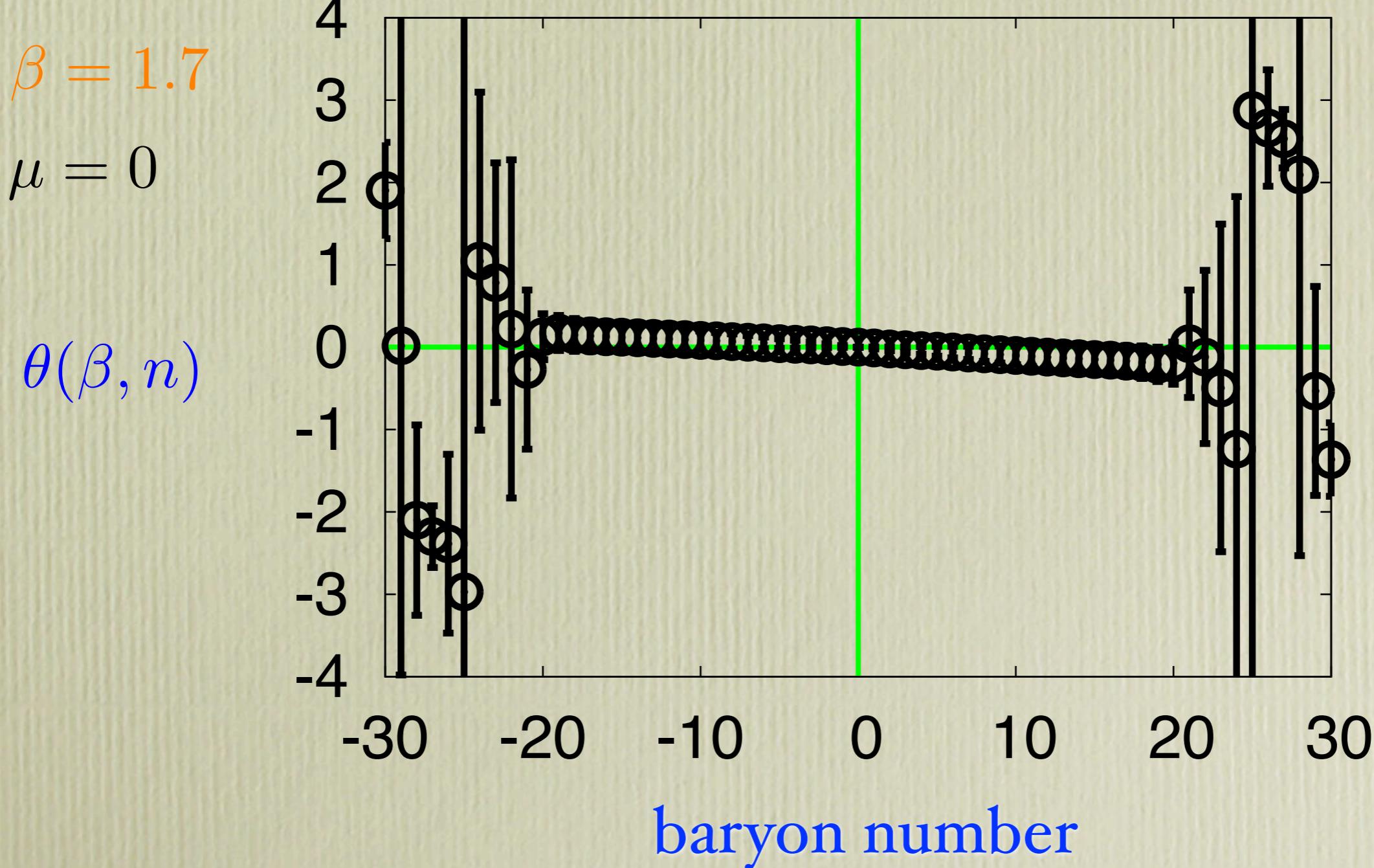
$\theta(\beta, n)$



Numerical results Phase($Z_c(n)$)

Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$



Numerical results Phase($Z_C(n)$)

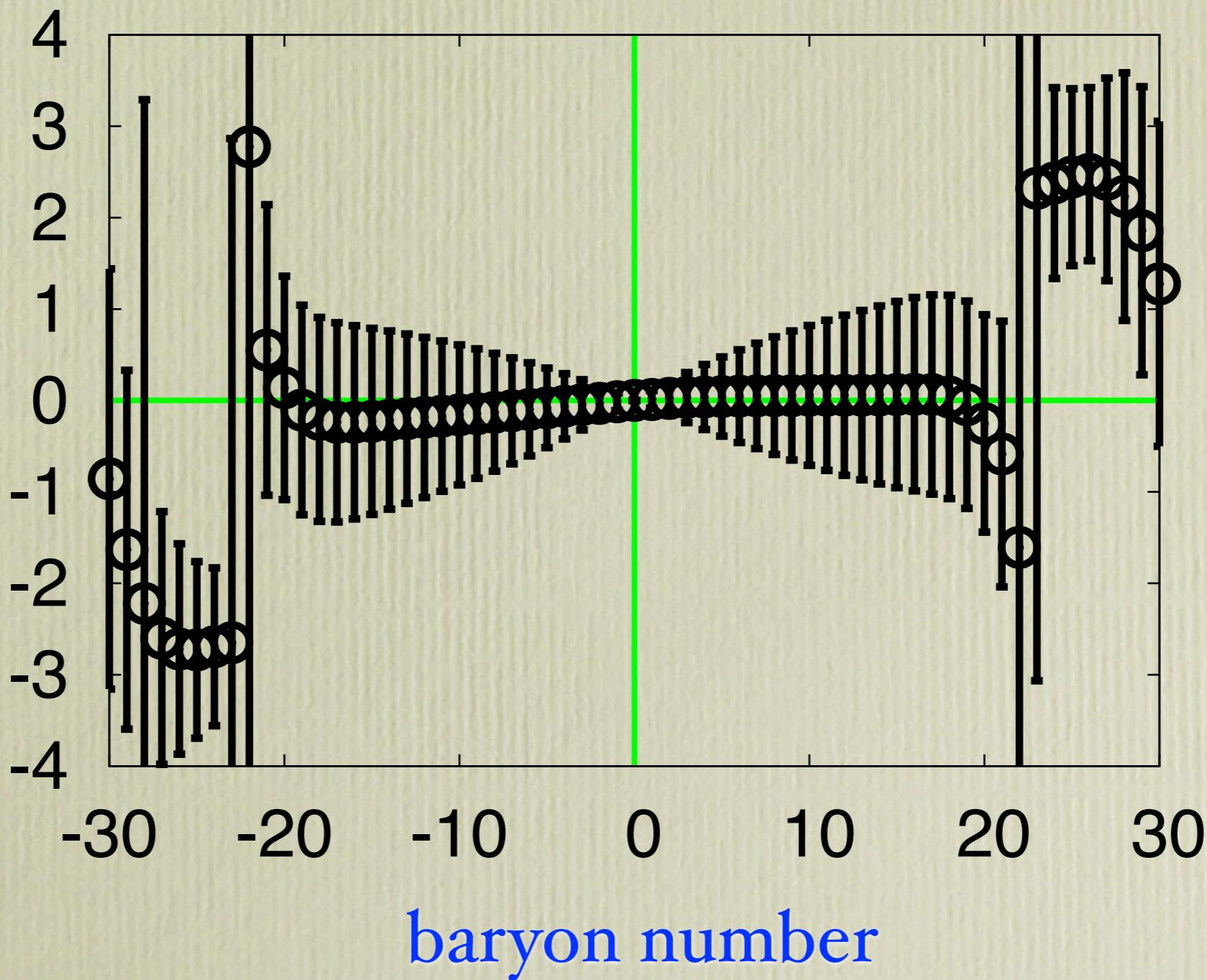
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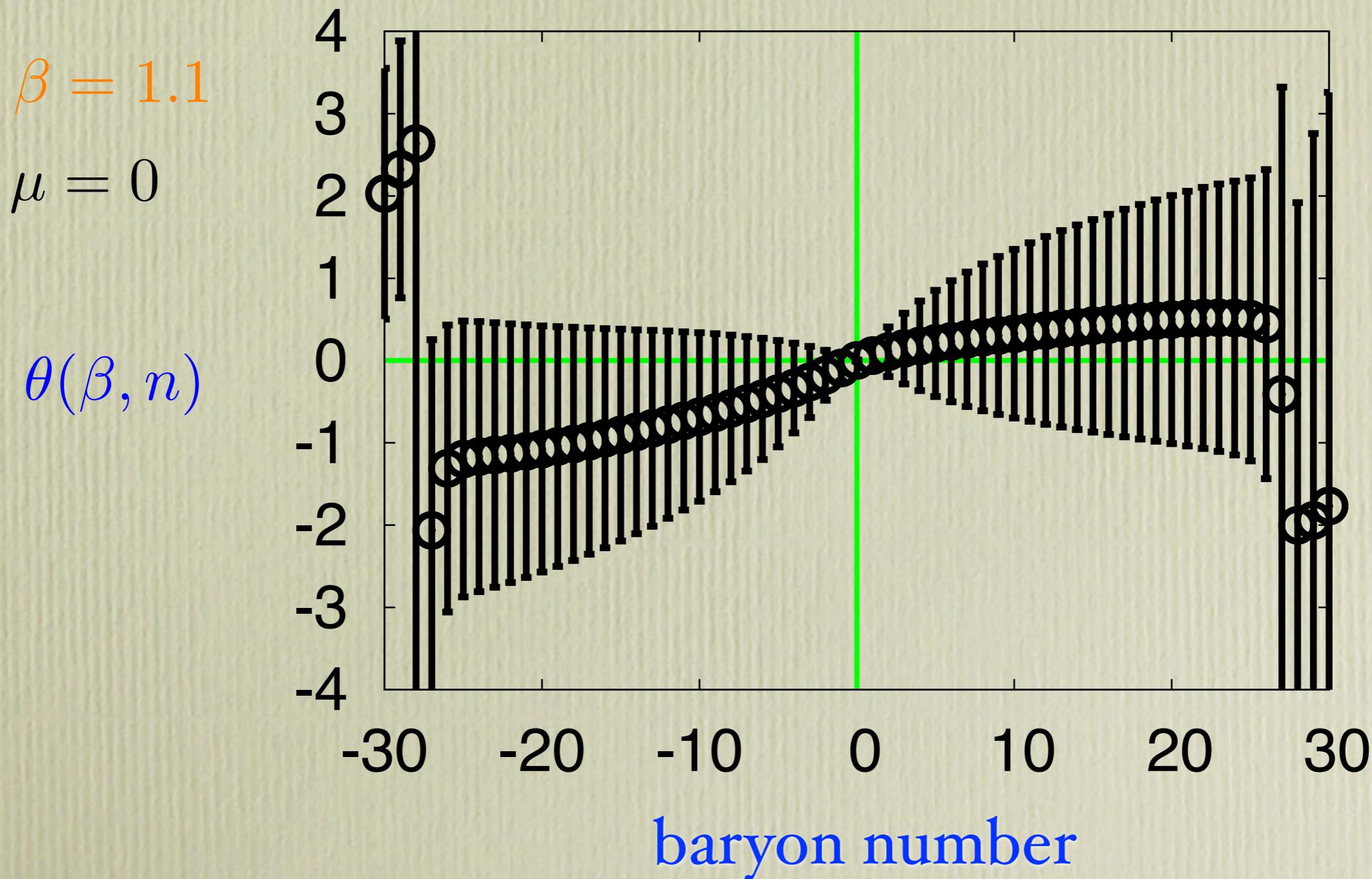
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Plan of the talk

- ✓ 1. Introduction
- ✓ 2. Winding number expansion
- ✓ 3. Numerical setup
- ✓ 4. Numerical results
- 5. Hadronic observables
- 6. Conclusion

Hadronic observables

Fugacity expansion of EV of GC observables

$$\langle \hat{O} \rangle_G(\beta, \mu, V) = \frac{\text{Tr} [\hat{O} \exp(-\beta (\hat{H} - \mu \hat{N}))]}{\text{Tr} [\exp(-\beta (\hat{H} - \mu \hat{N}))]}$$

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path integral formalism

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path integral formalism  re-weighting technique

$$\left\langle O(D_W(\mu)) \frac{\text{Det} D_W(\mu)}{\text{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$$

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$$O_n = \oint \frac{d\xi}{2\pi i} \xi^{-n-1} \left\langle O(D_W(\xi)) \frac{\text{Det} D_W(\xi)}{\text{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$$

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To see explicit functional form in ξ

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To see explicit functional form in ξ Use HPE

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$$\bar{\psi}\psi = -\text{tr} \left(\frac{1}{D_W} \right) = -\text{tr} \left(\frac{1}{1 - \kappa Q} \right) = \sum_{m=0}^{\infty} \kappa^m \text{tr} Q^m$$

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resummation

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Hadronic observables

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resummation

$$\sum_{n=-\infty}^{\infty} S_n(U) \xi^n$$
$$\exp \left(\sum_{n=-\infty}^{\infty} W_n(U) \xi^n \right)$$

$\text{Det} D_W$

resummation

Hadronic observables

$$O_n = \sum_E \left\langle E, n \left| \hat{O} e^{-\beta \hat{H}} \right| E, n \right\rangle$$

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EV of canonical ensemble

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$$\longrightarrow \langle \hat{O} \rangle_G(\beta, \mu, V) = \frac{O(\mu)}{Z(\mu)}$$

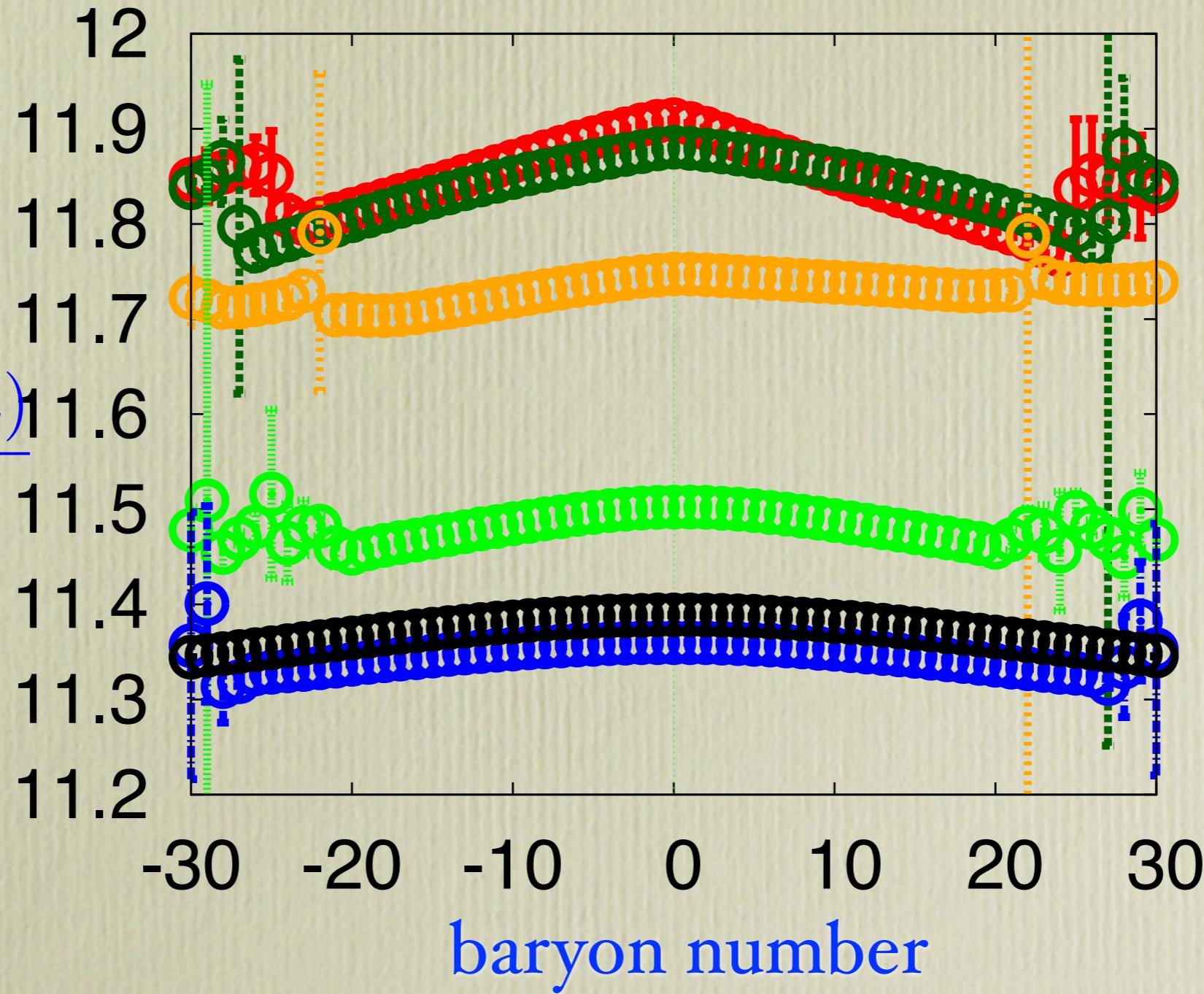
Hadronic observables

Chiral condensate in canonical ensemble

No renormalization! No subtraction! Sorry...

Preliminary!

$$\frac{\sum \langle \bar{\psi} \psi \rangle_C(\beta, n)}{V}$$



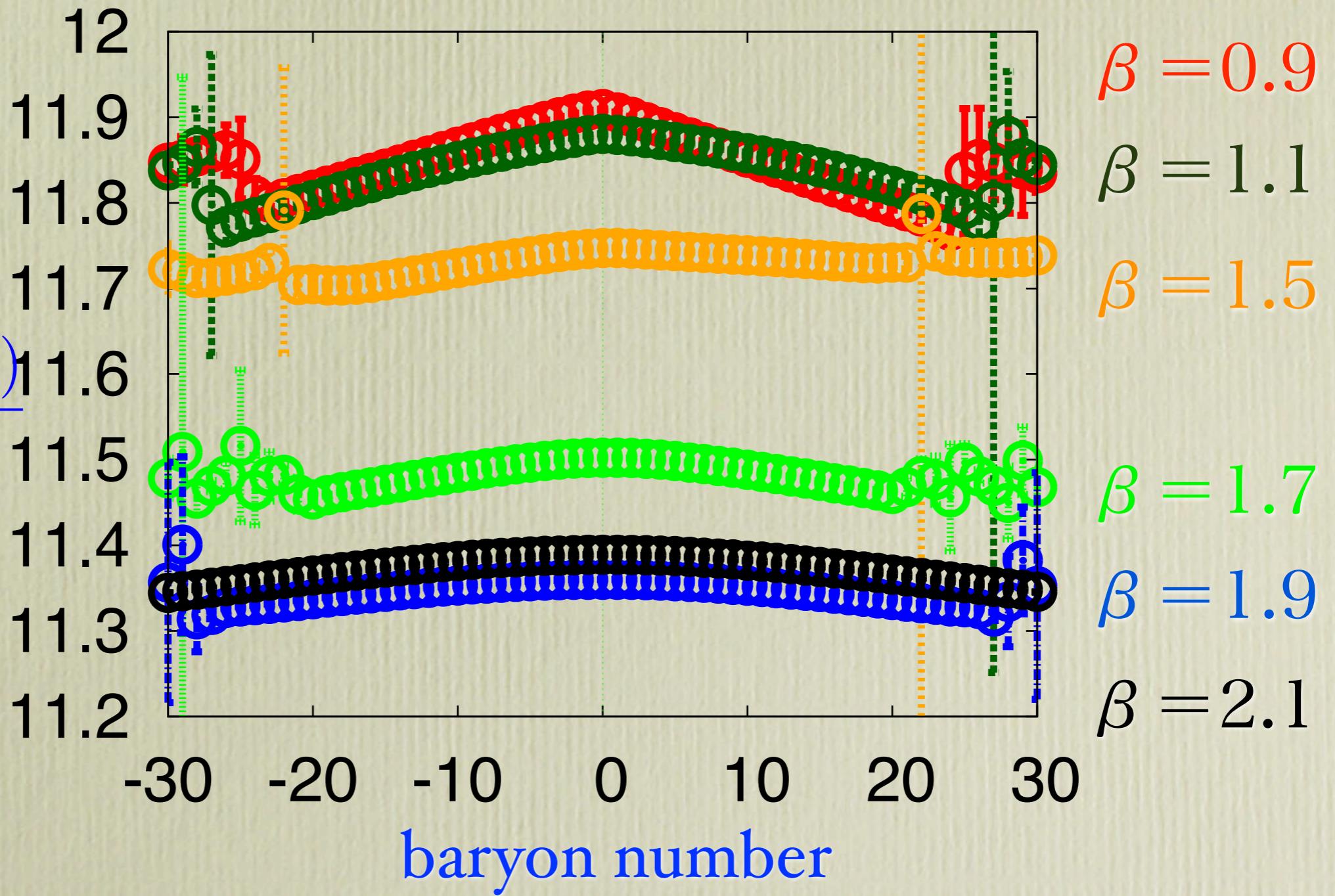
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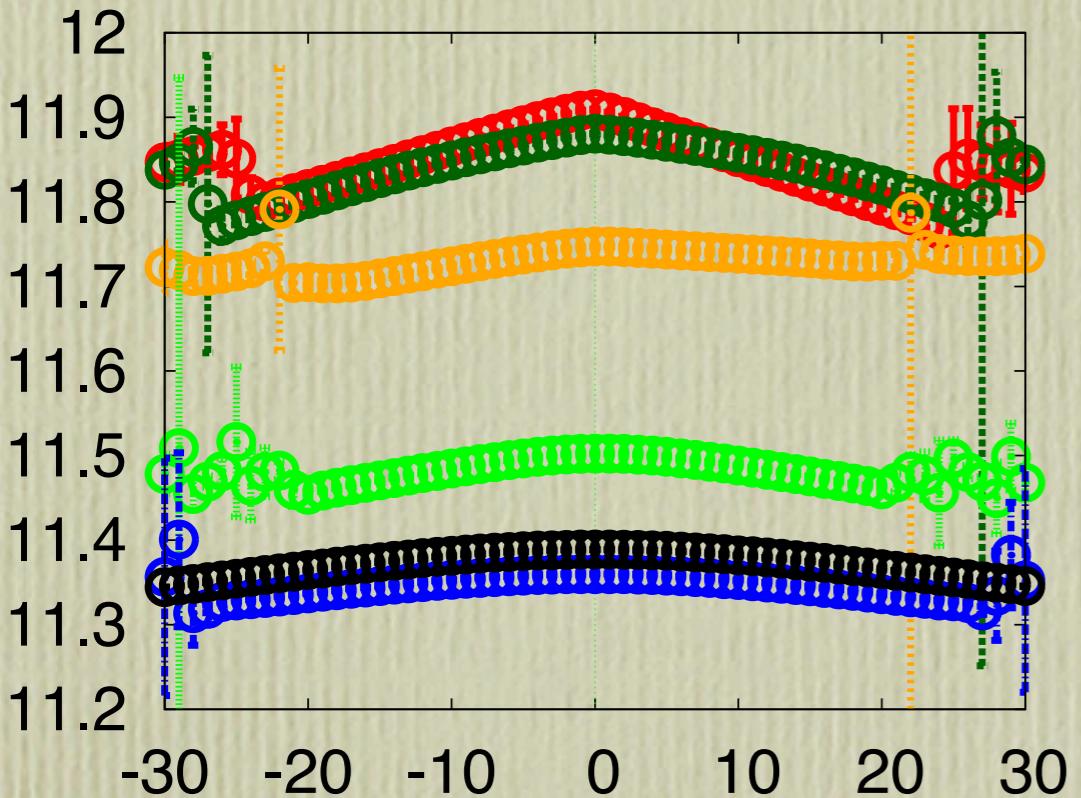
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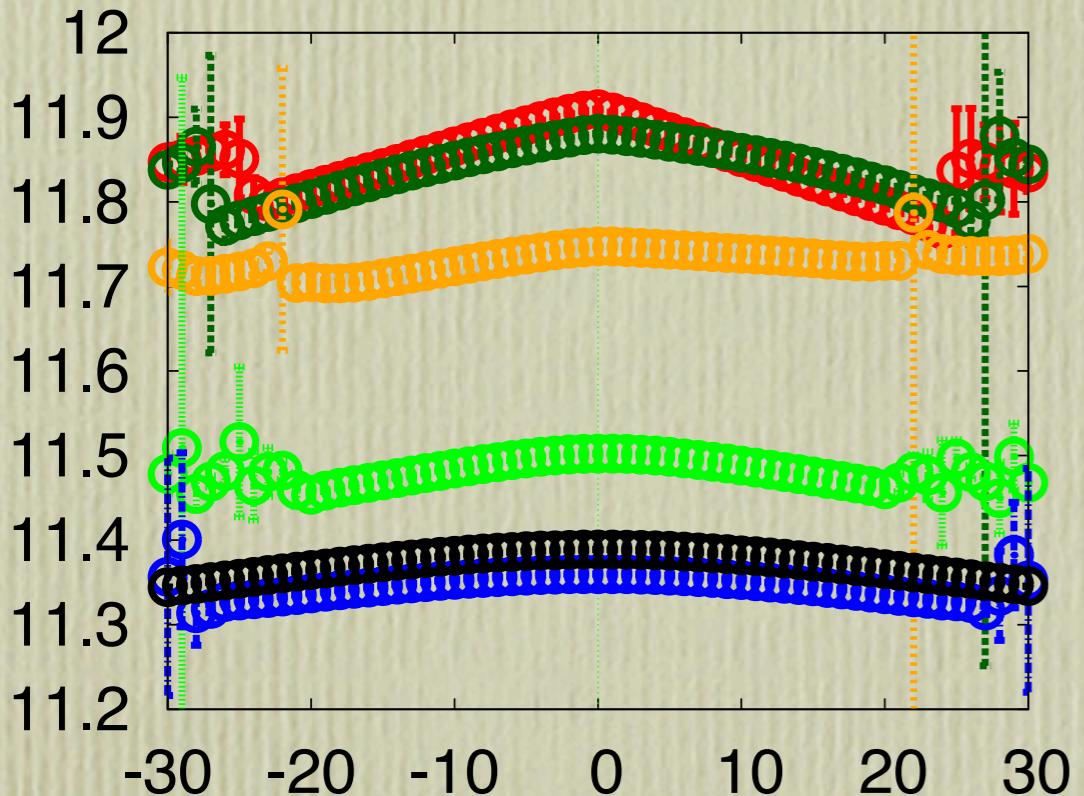
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Hadronic observables

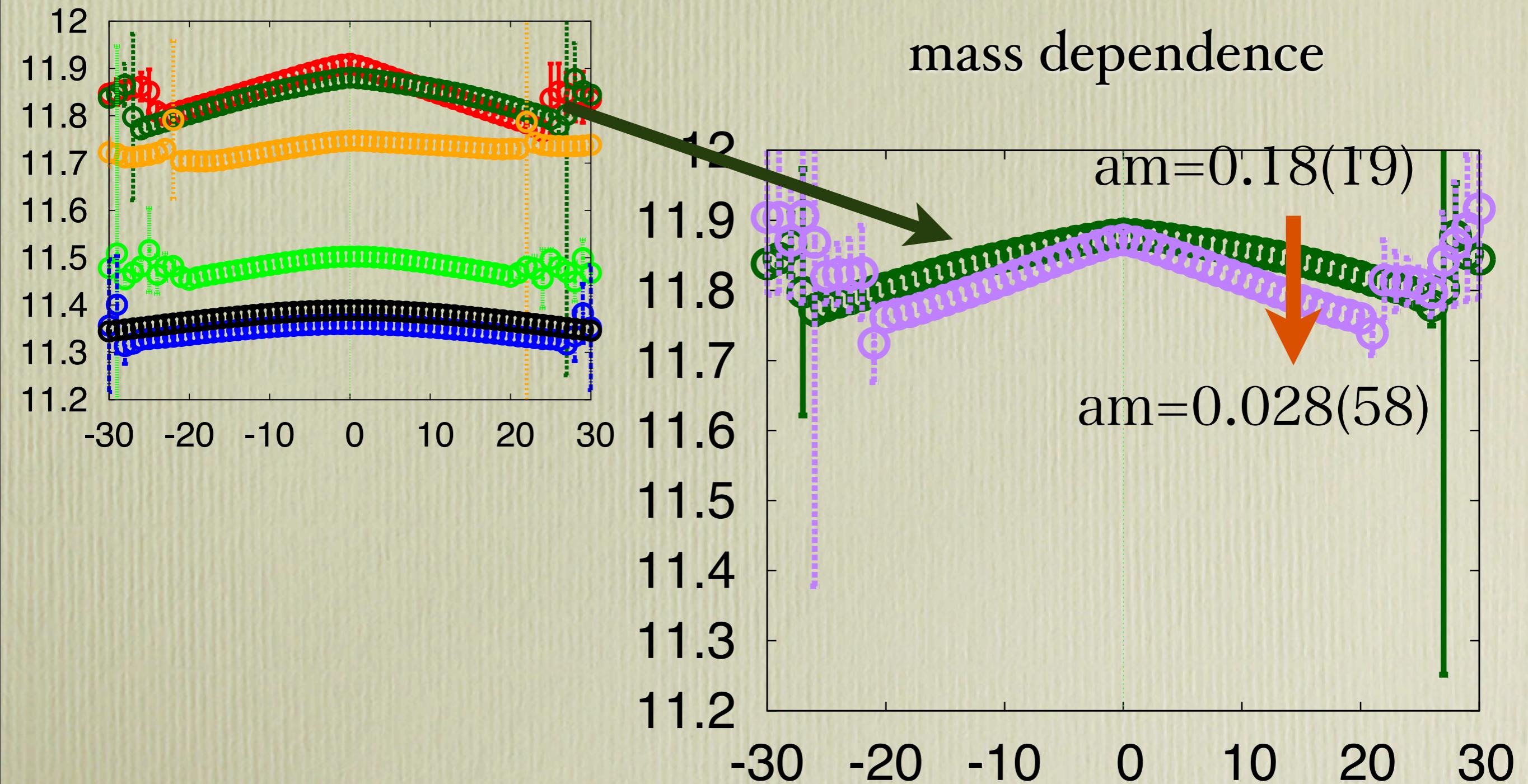
Chiral condensate in canonical ensemble



mass dependence

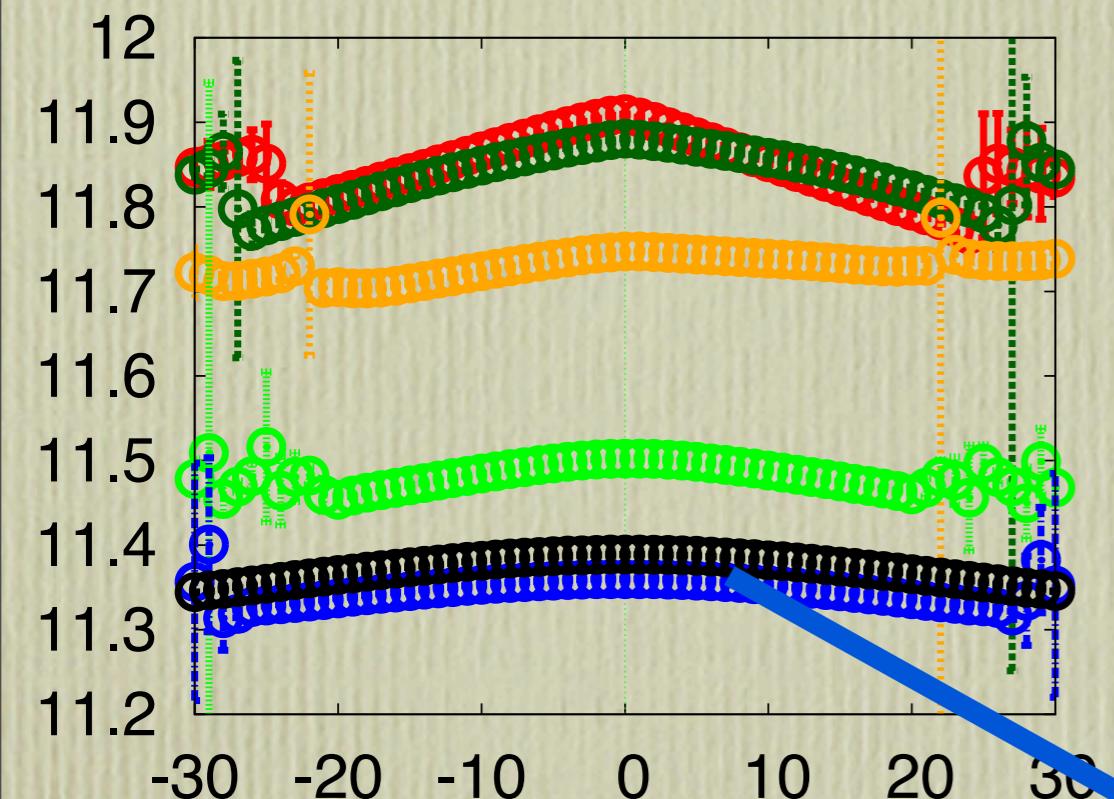
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Chiral condensate in canonical ensemble

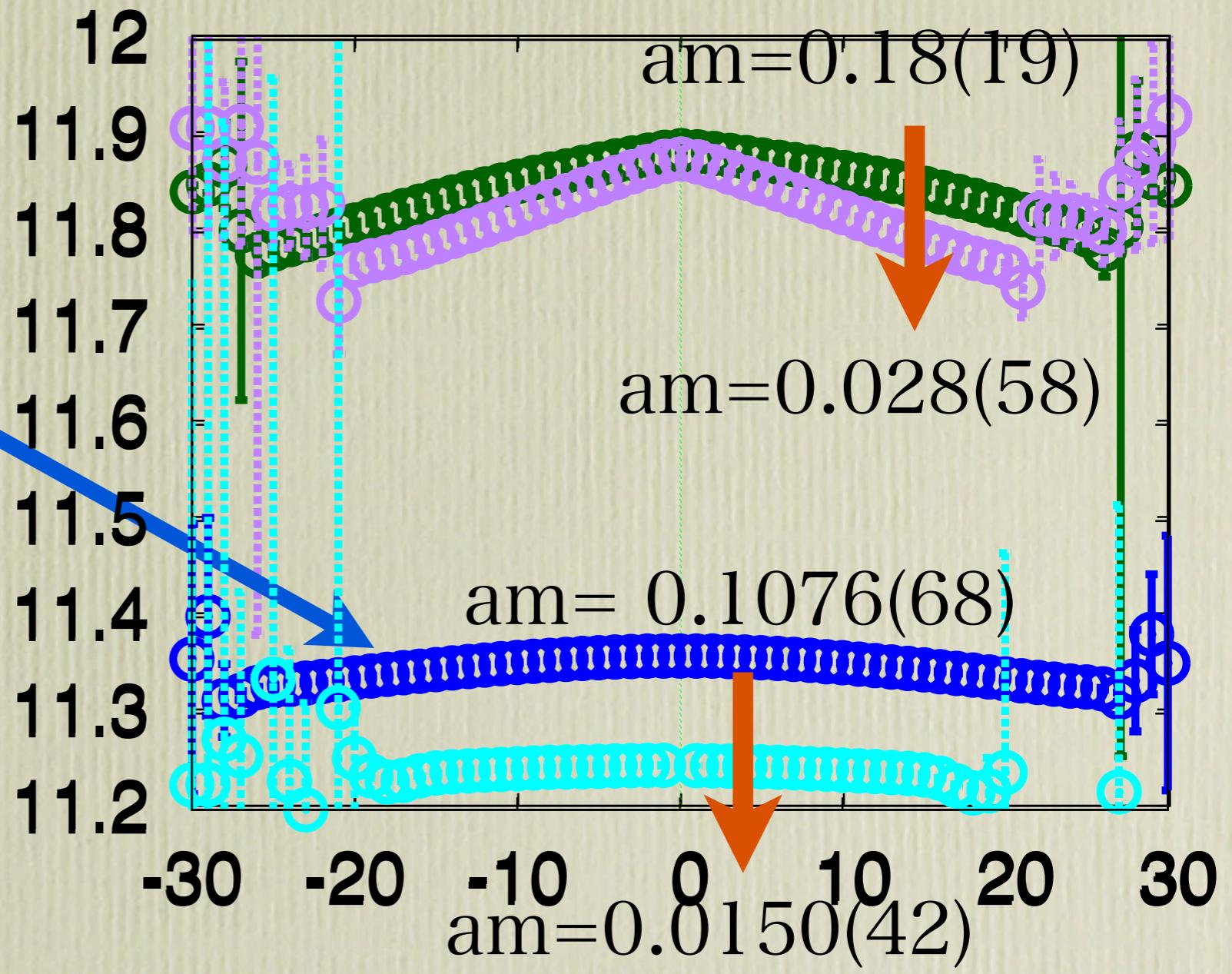


Hadronic observables

Chiral condensate in canonical ensemble



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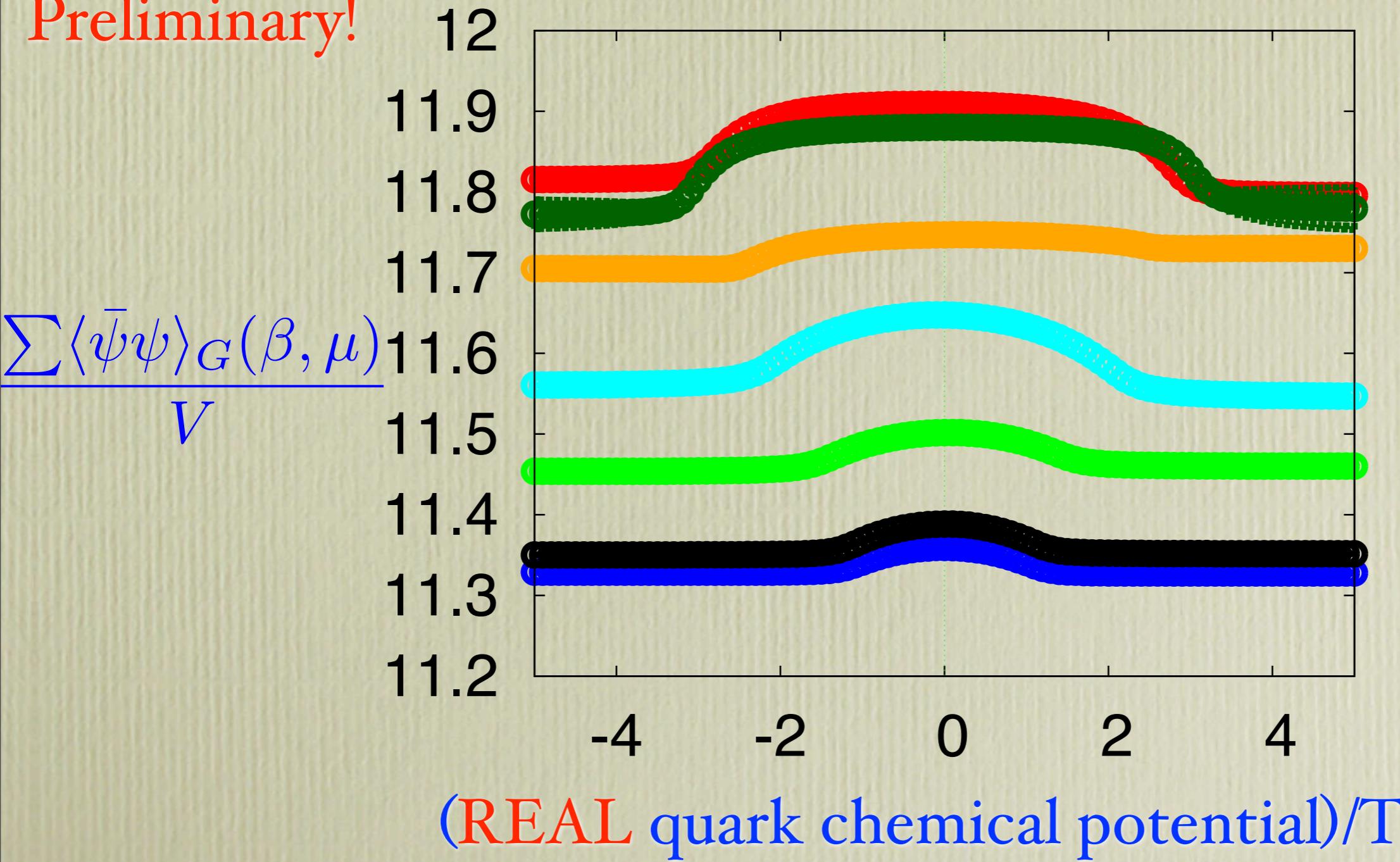


Hadronic observables

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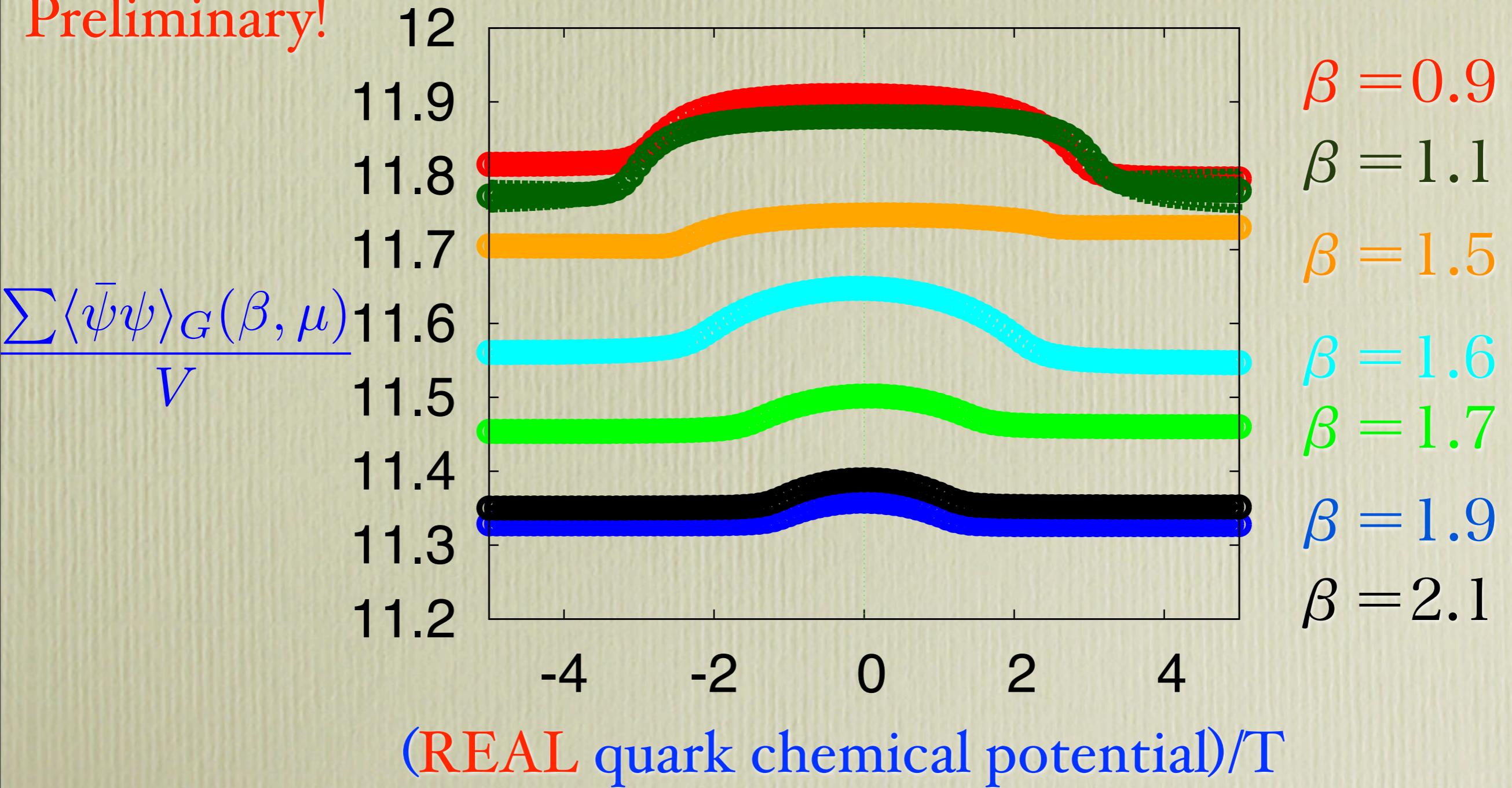


Hadronic observables

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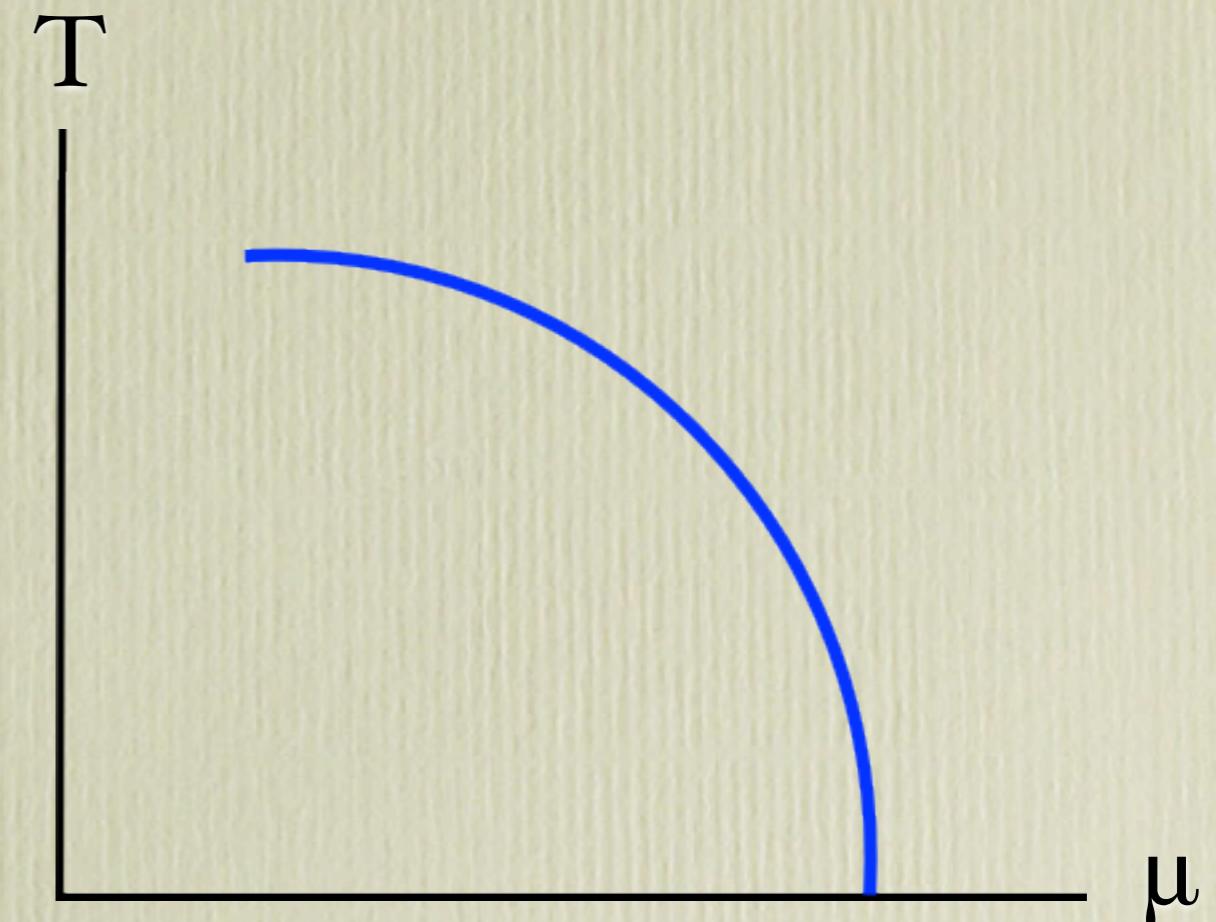
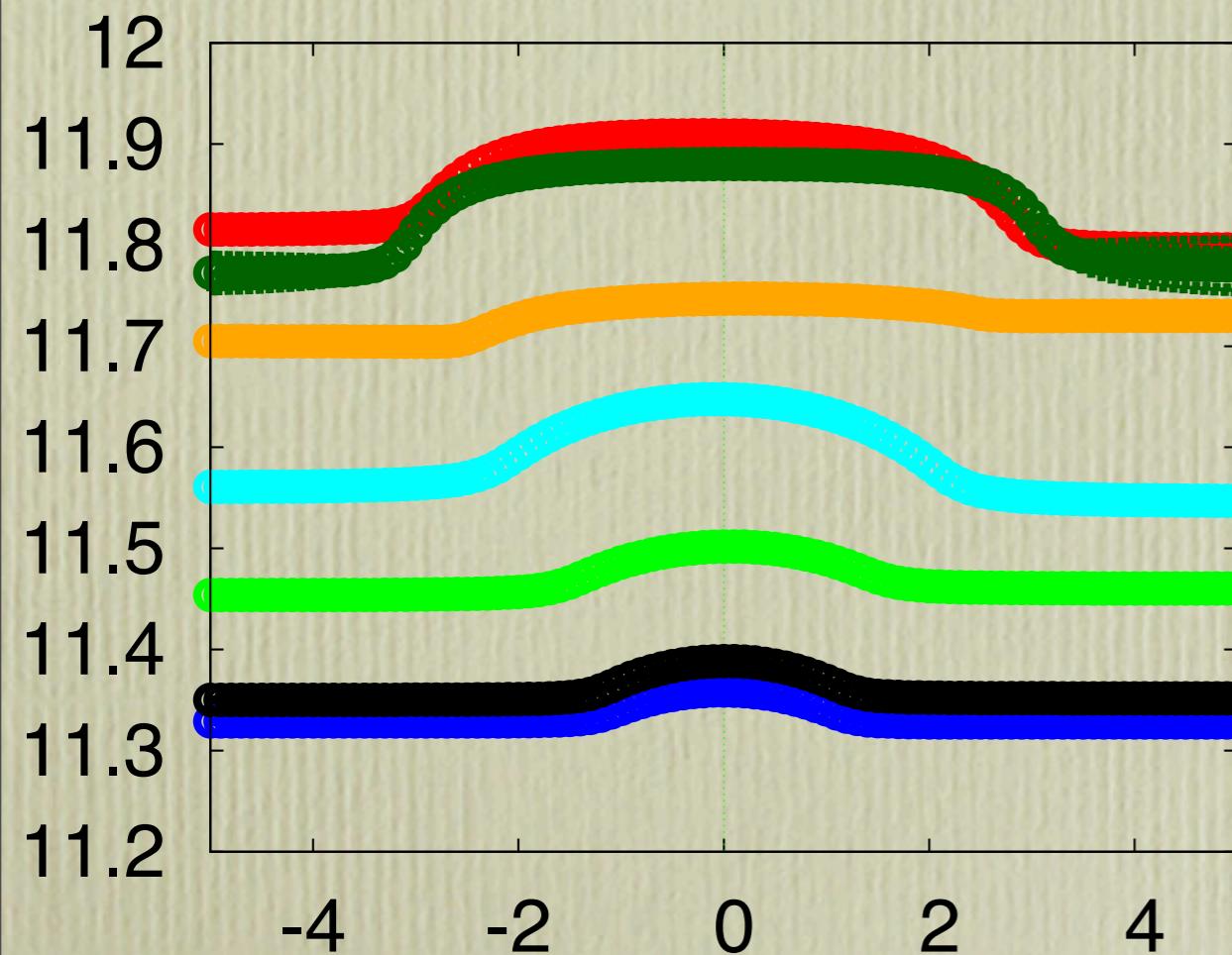
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Hadronic observables

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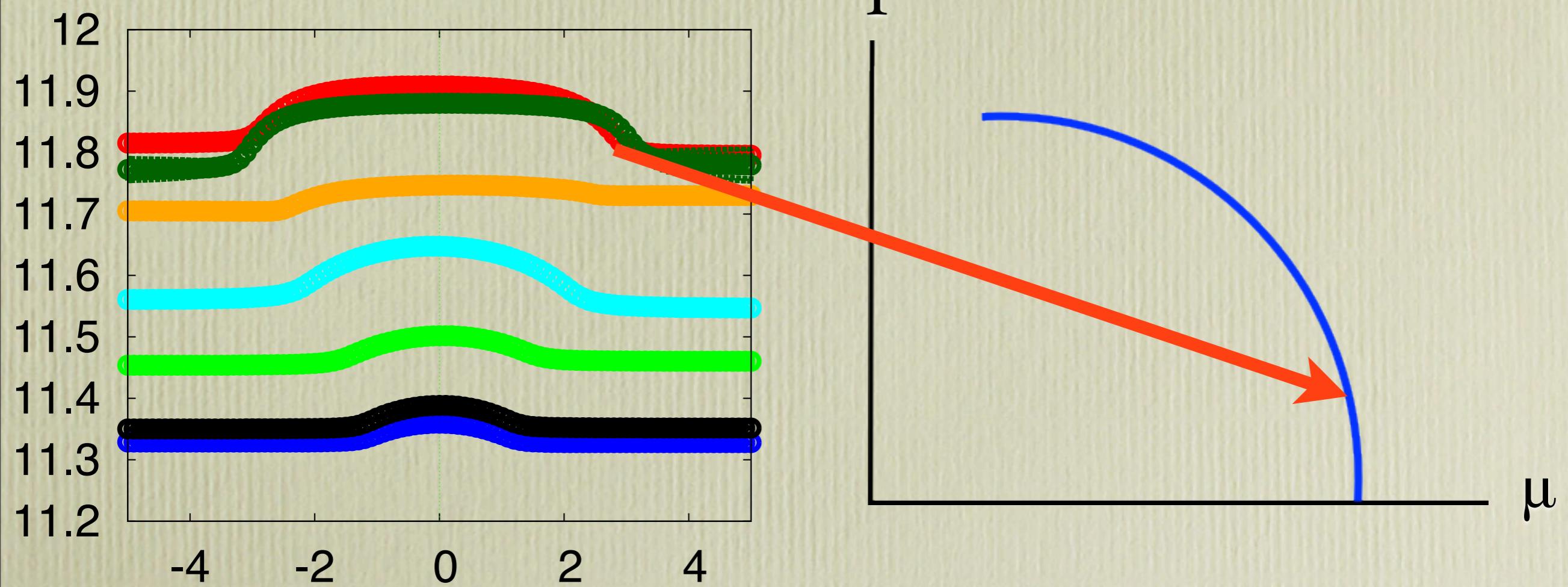
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Hadronic observables

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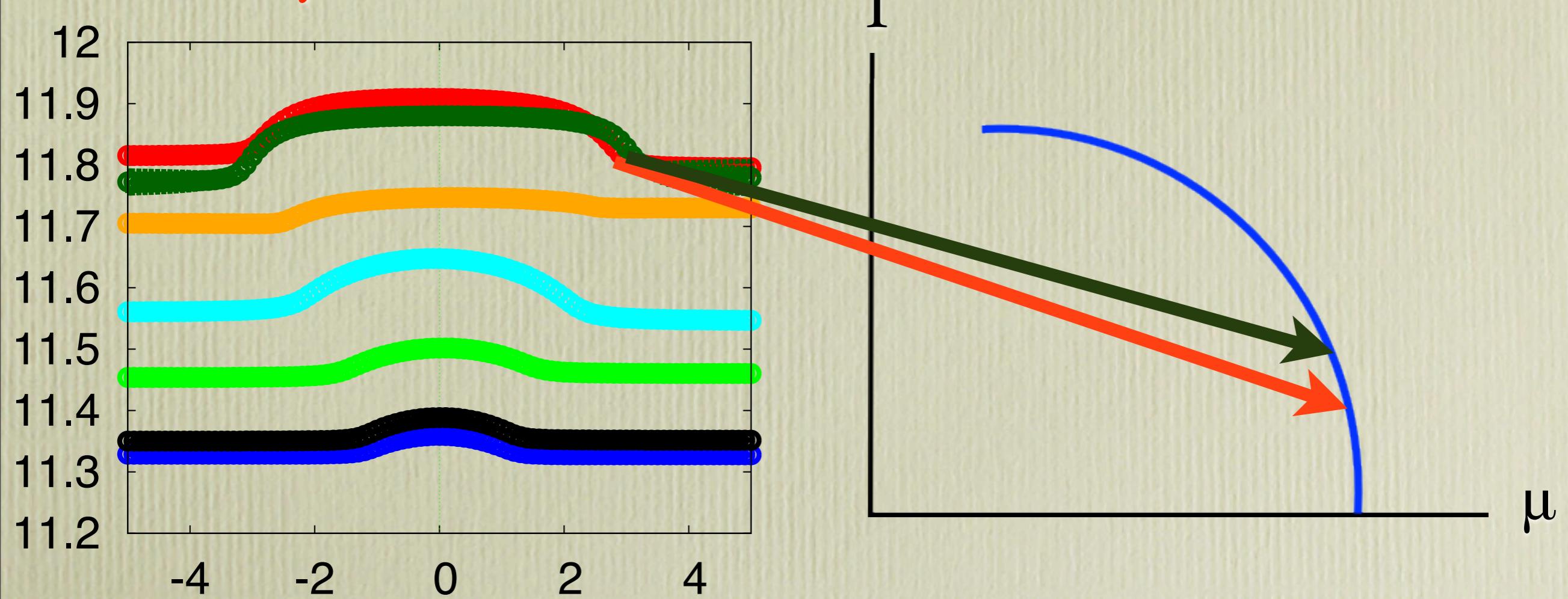
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Hadronic observables

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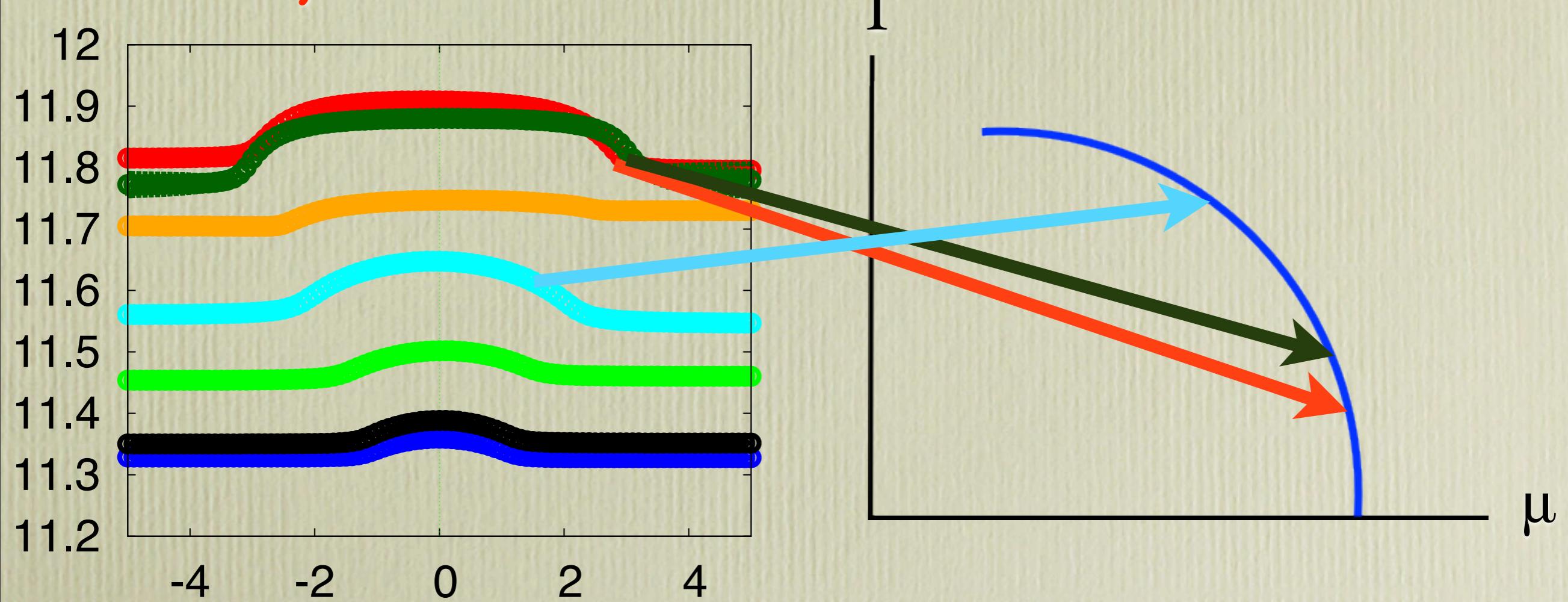
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Hadronic observables

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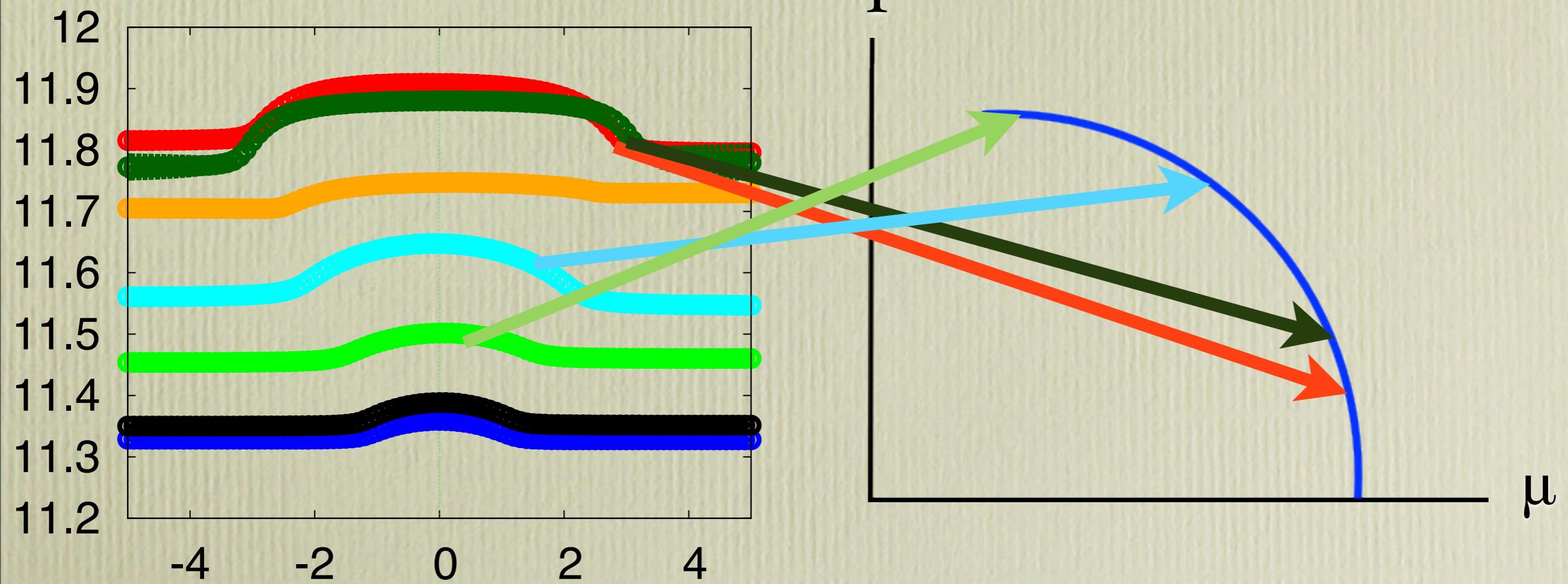
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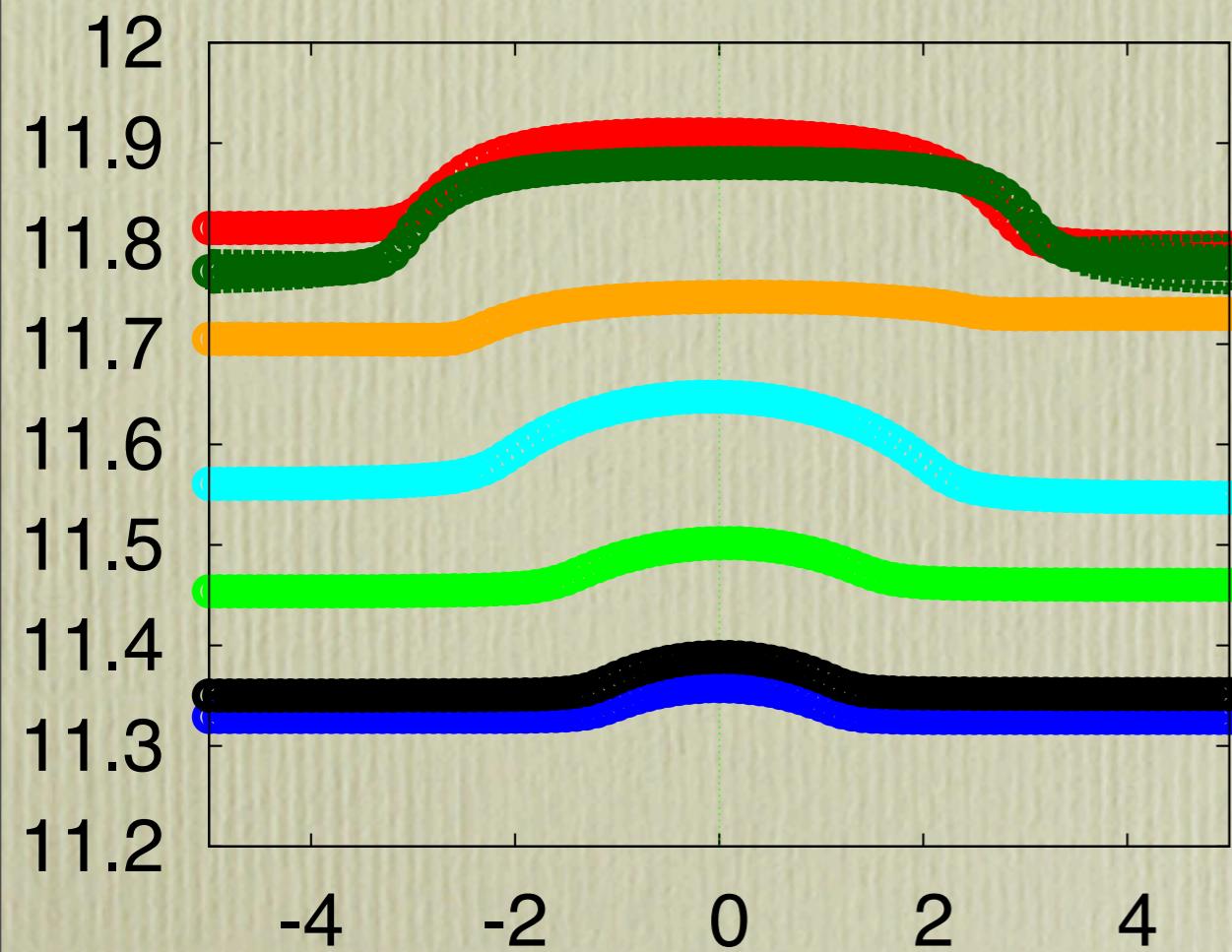


Hadronic observables

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Preliminary!

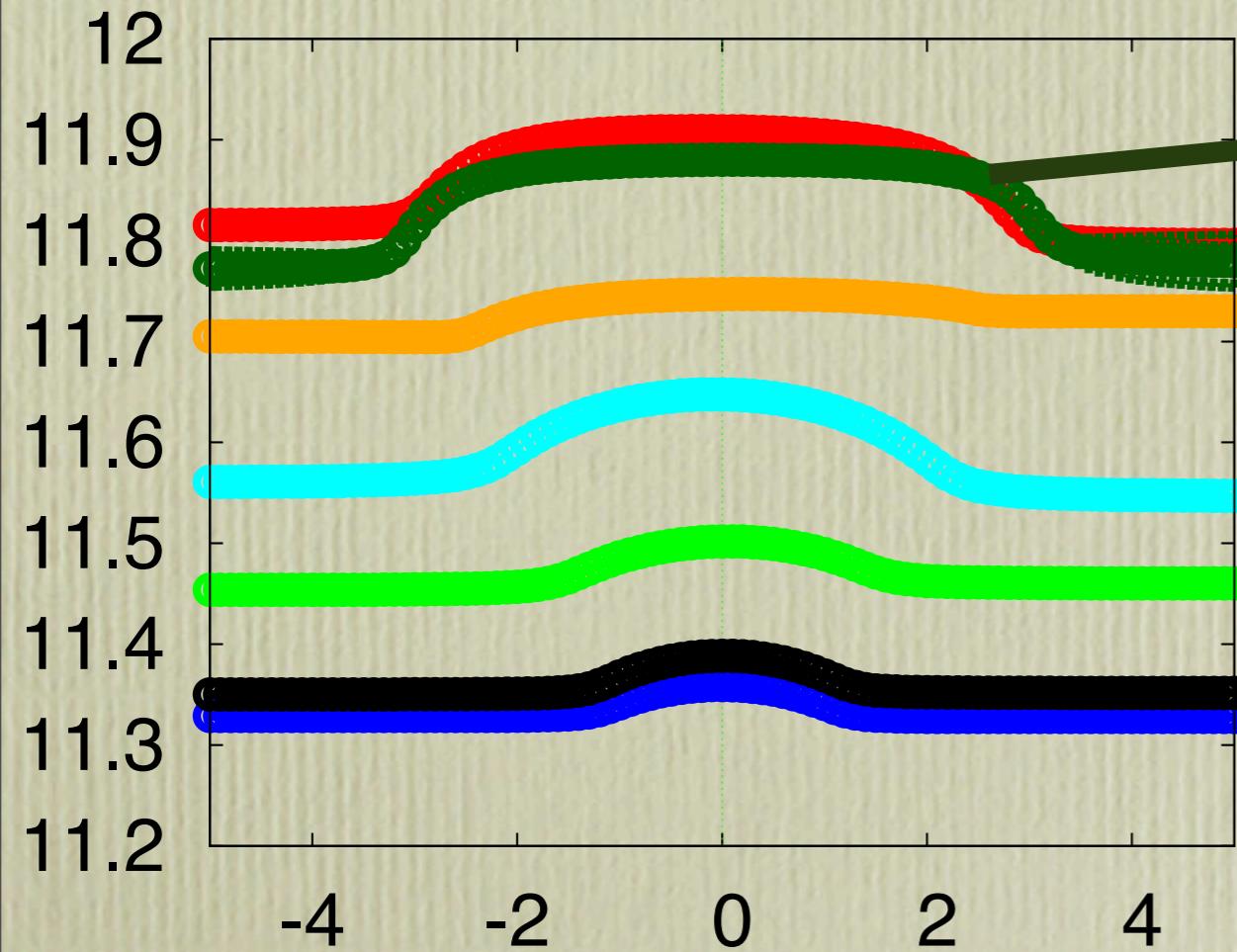
mass dependence



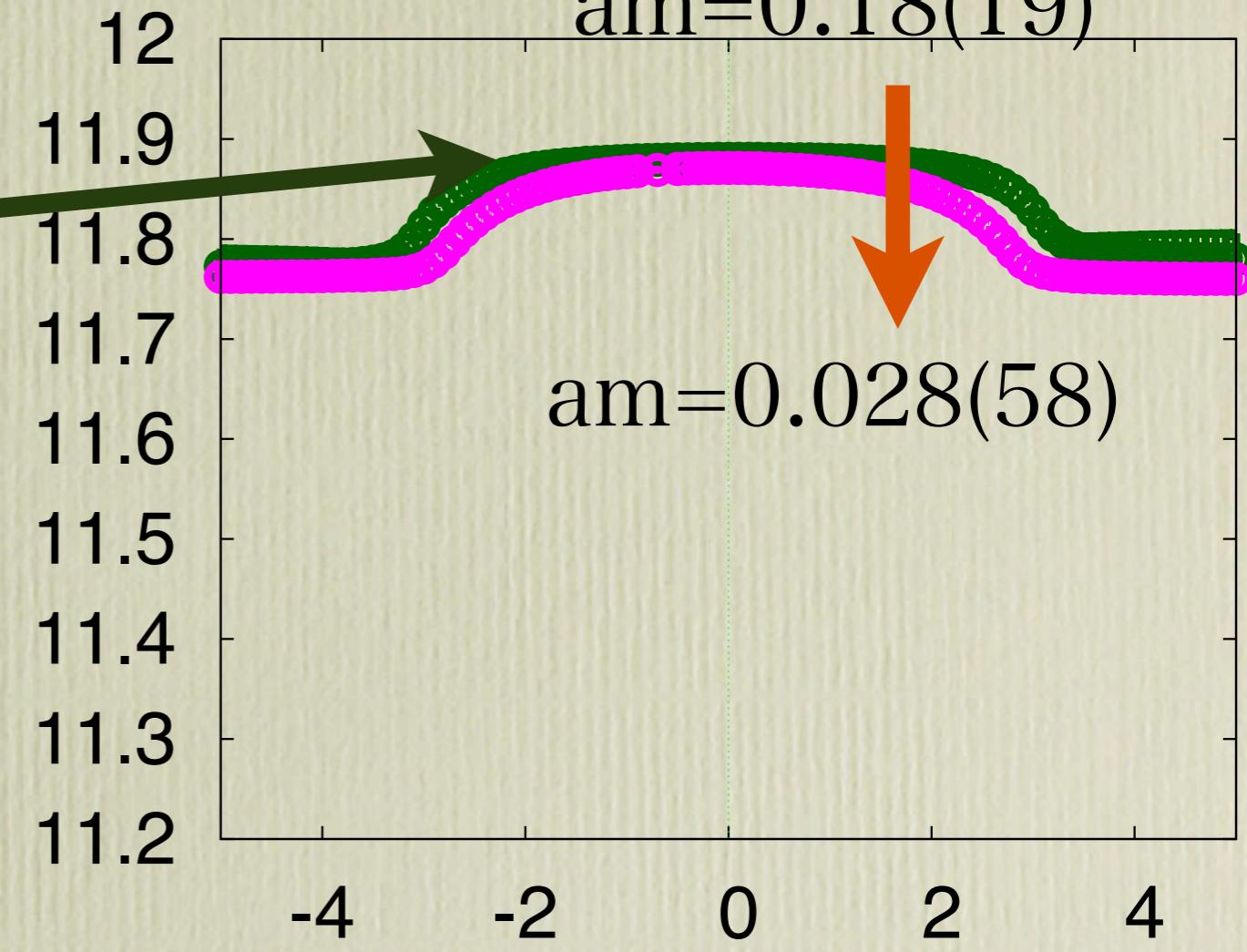
Hadronic observables

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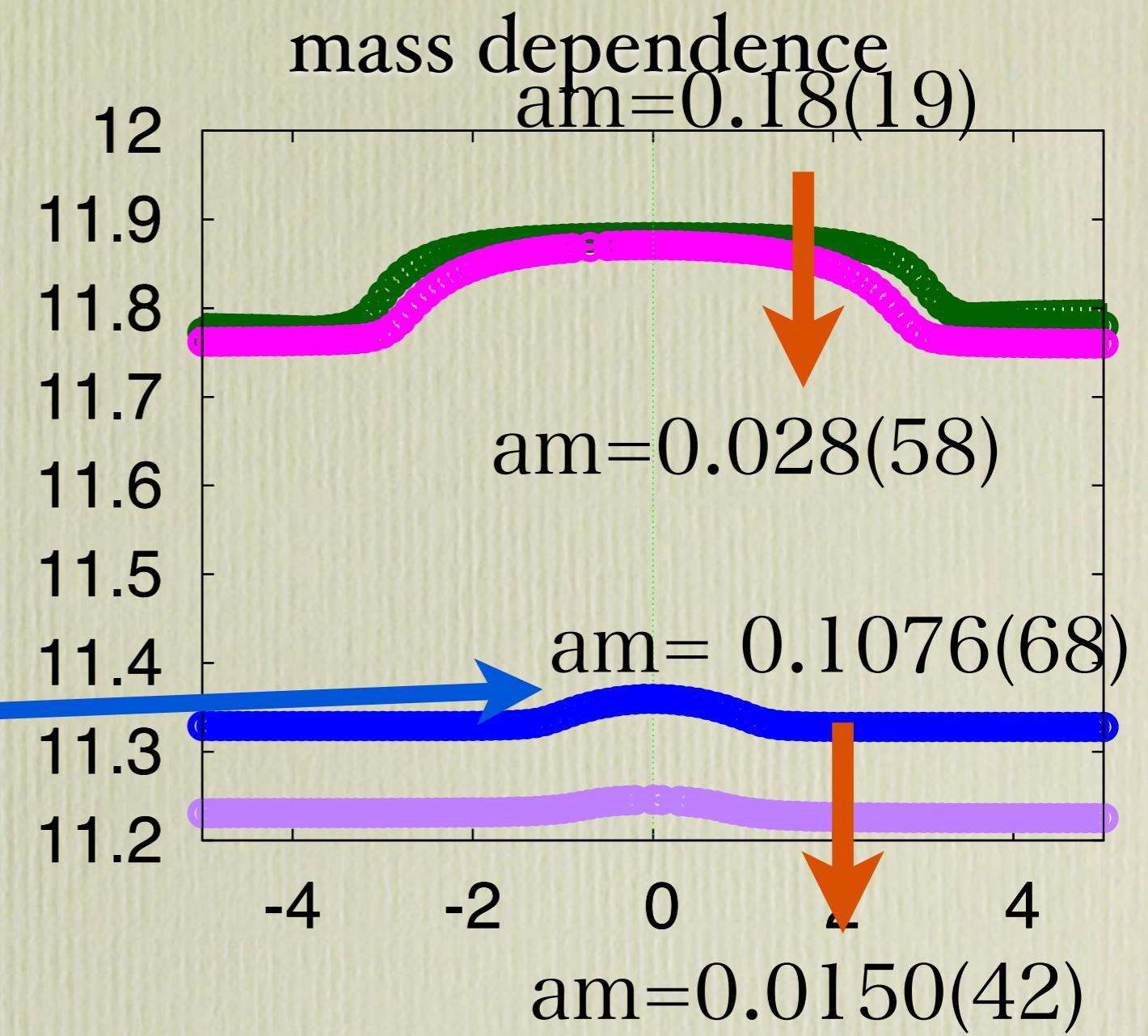
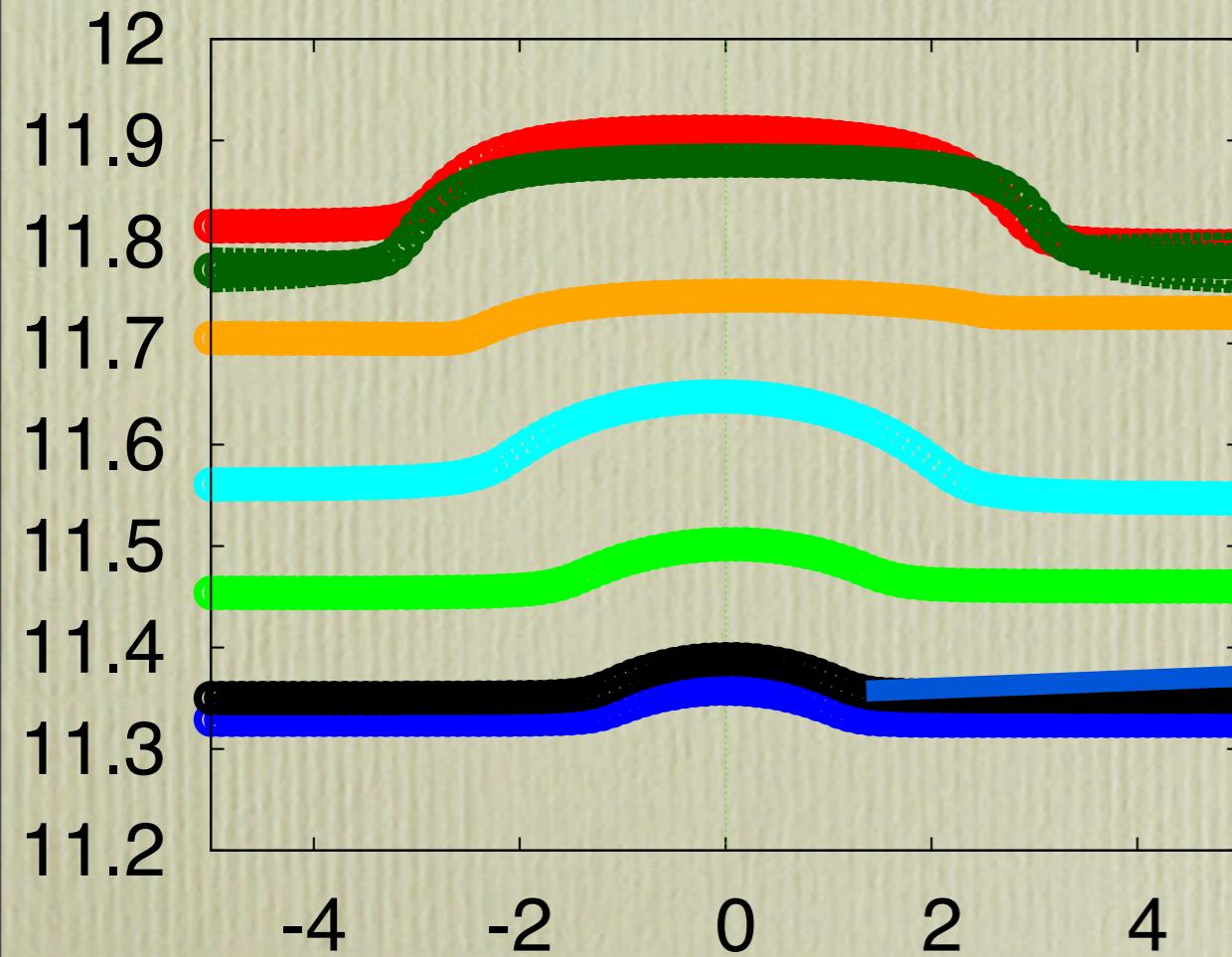
mass dependence
 $am=0.18(19)$



Hadronic observables

Chiral condensate in grand canonical ensemble

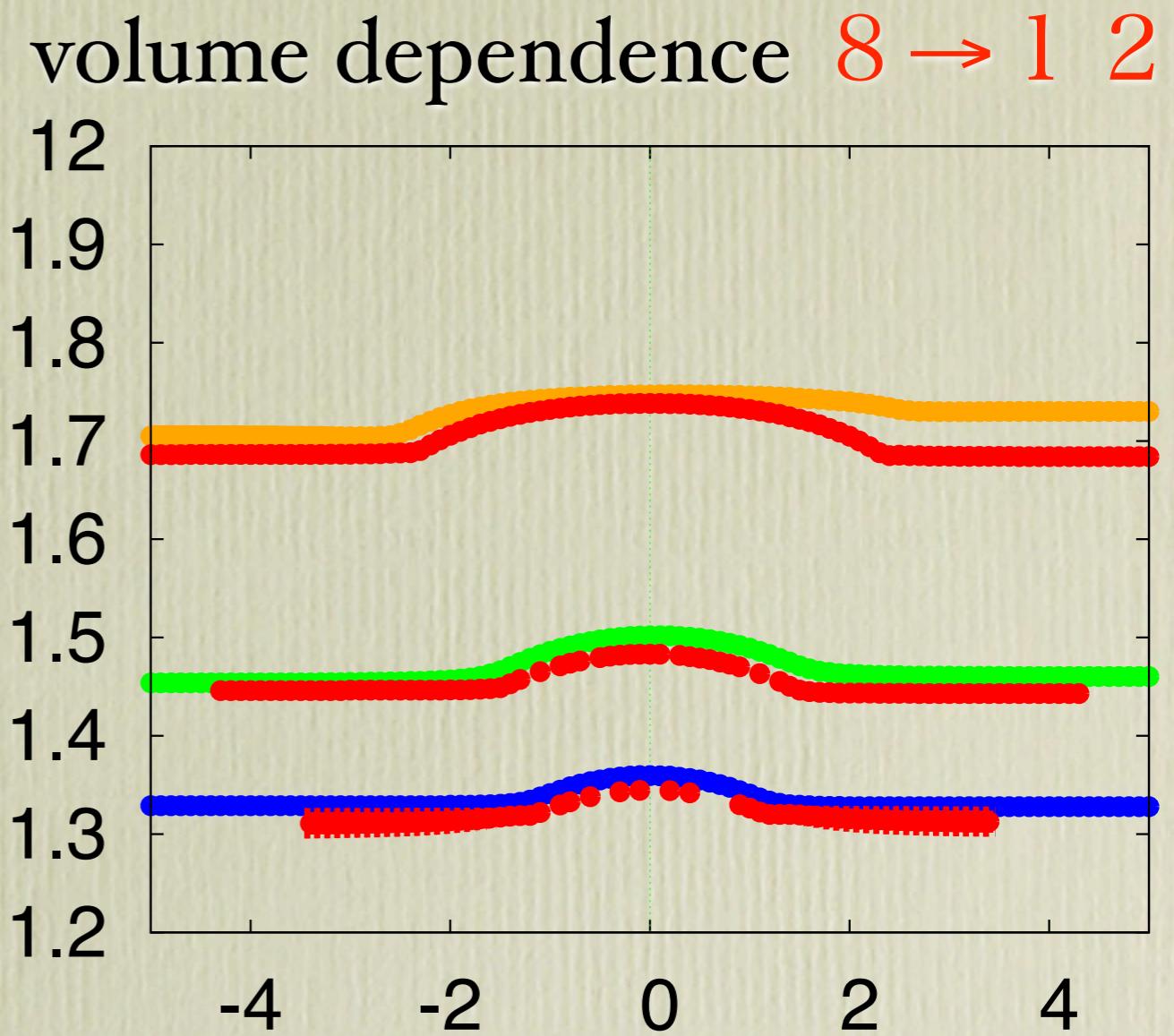
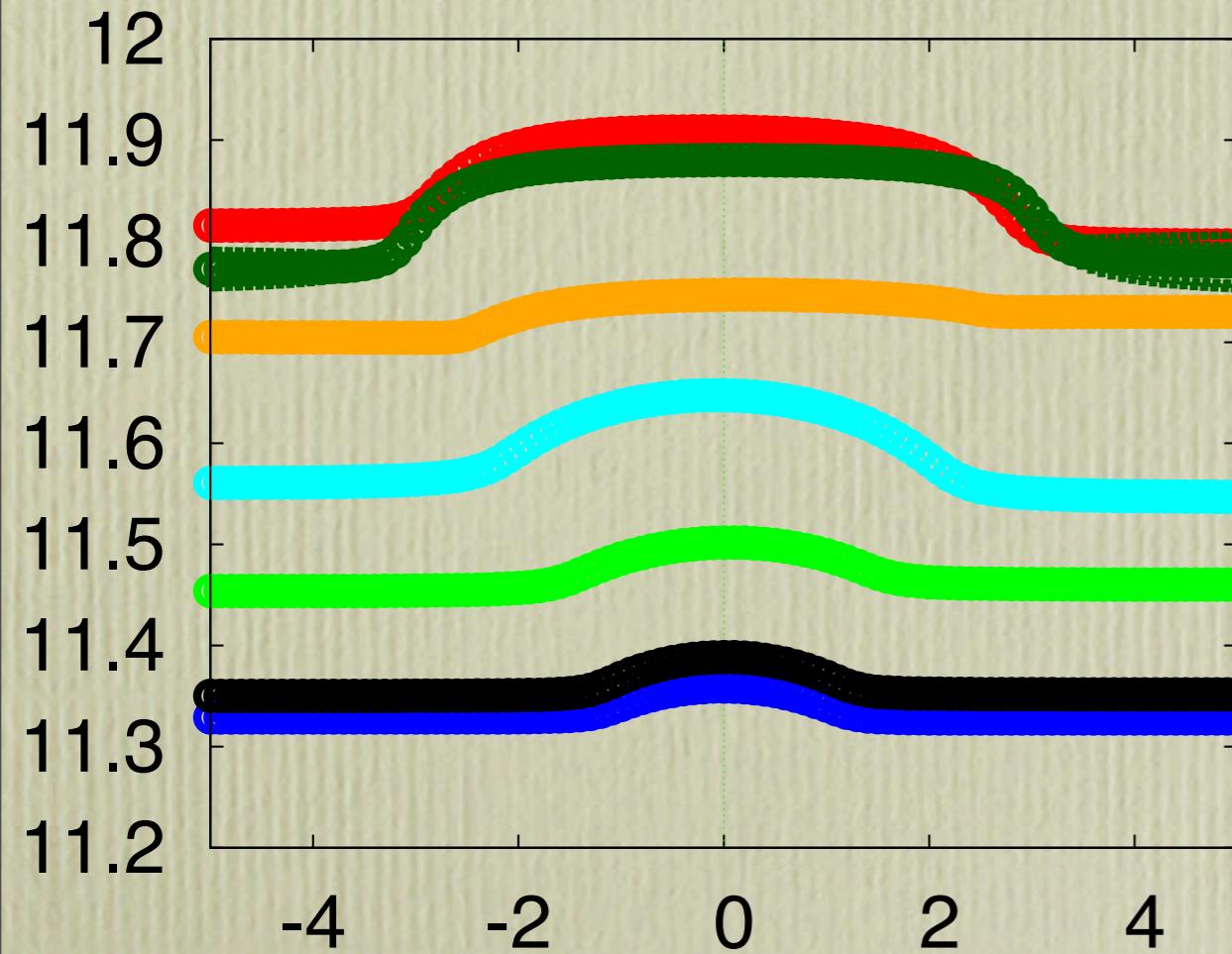
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Hadronic observables

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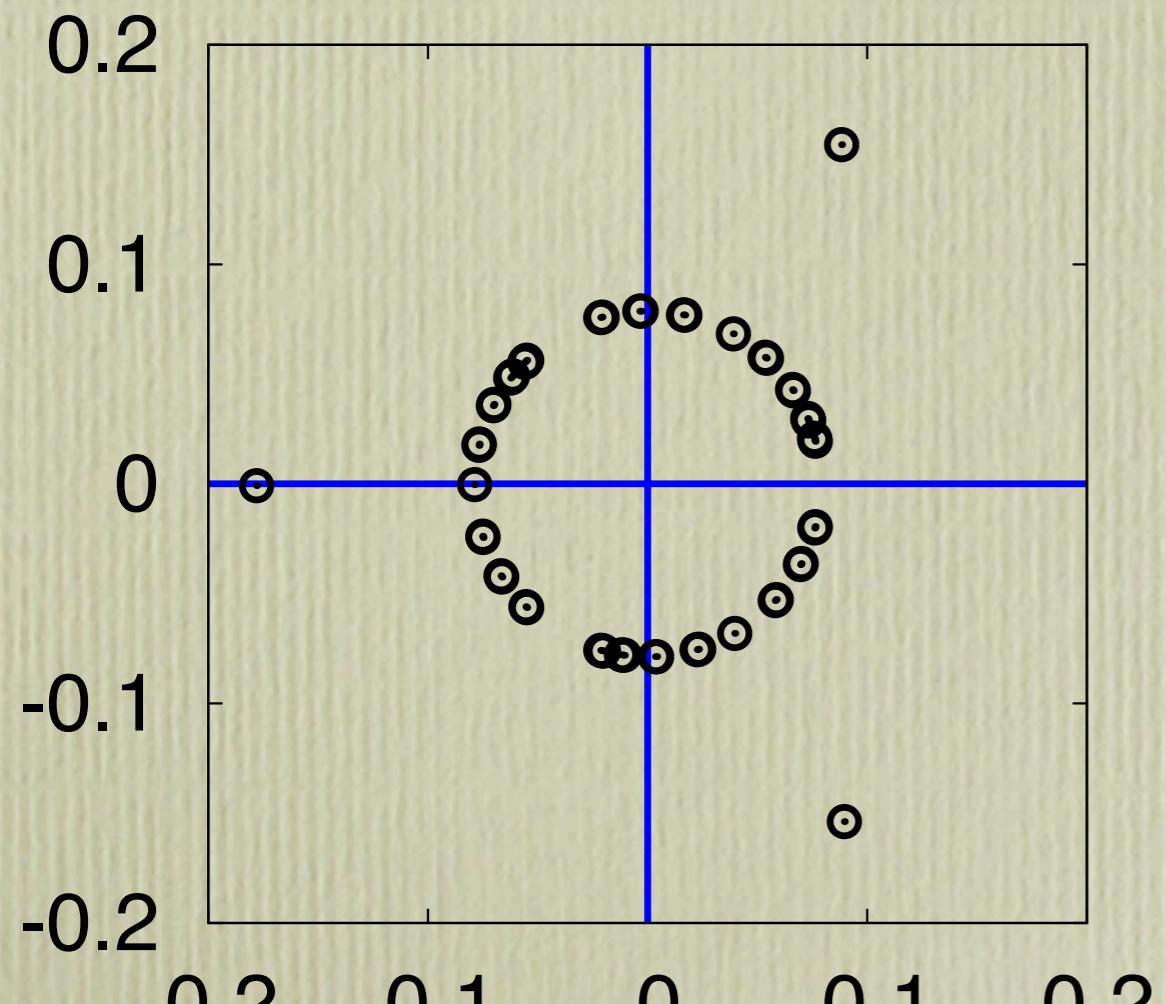
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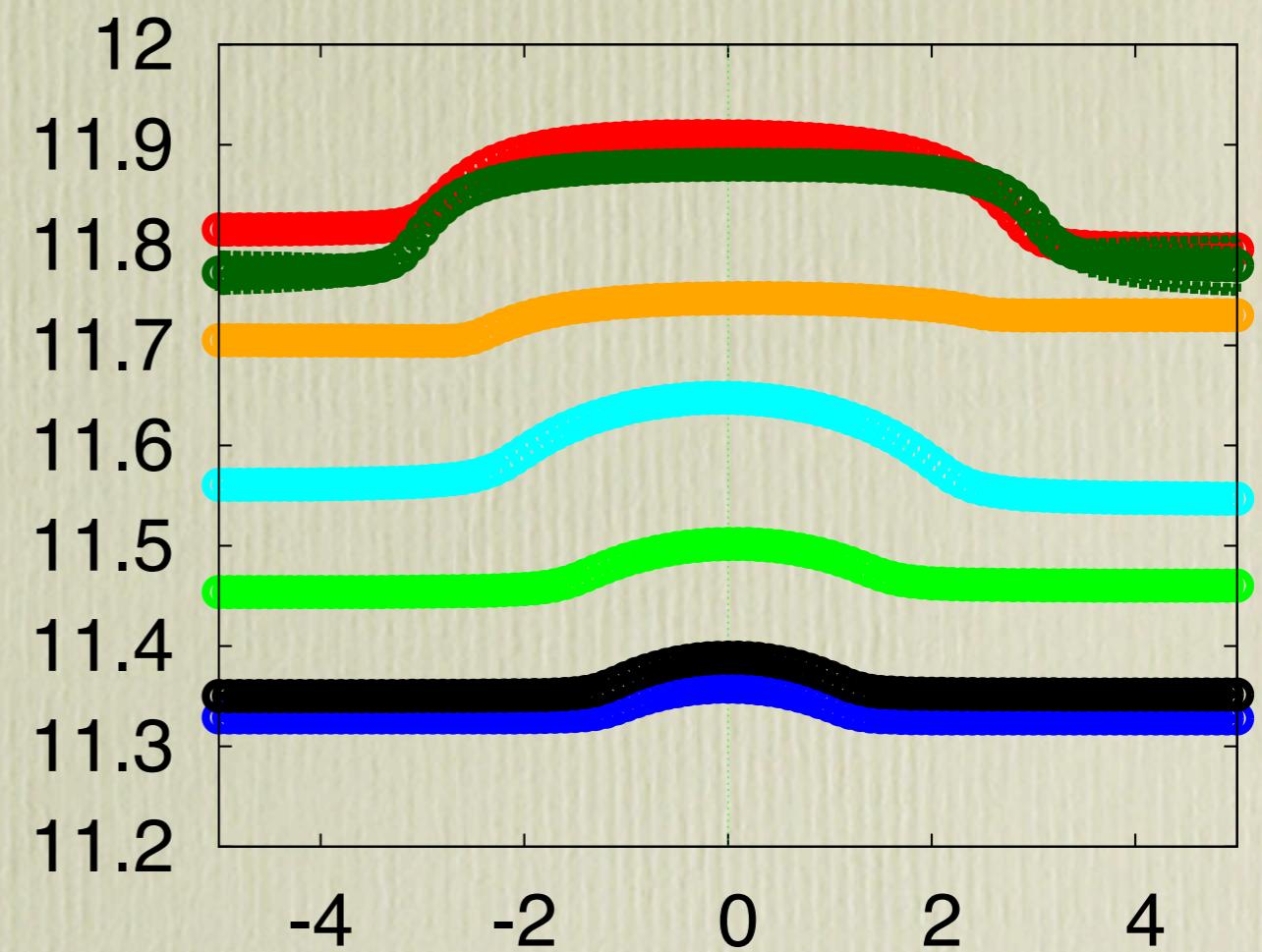
Hadronic observables

Chiral condensate in grand canonical ensemble

Lee-Yang zeros



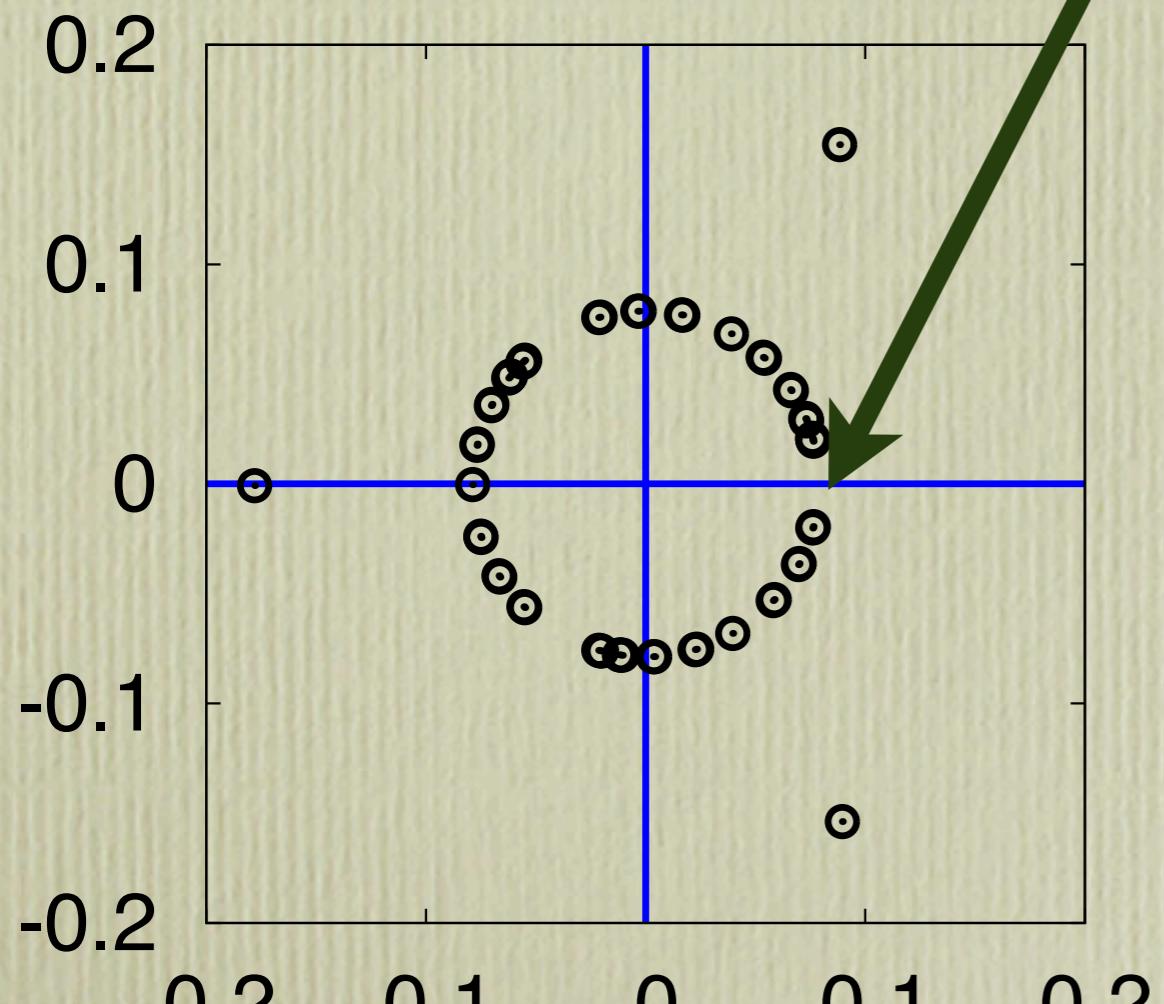
$$\beta = 1.1$$



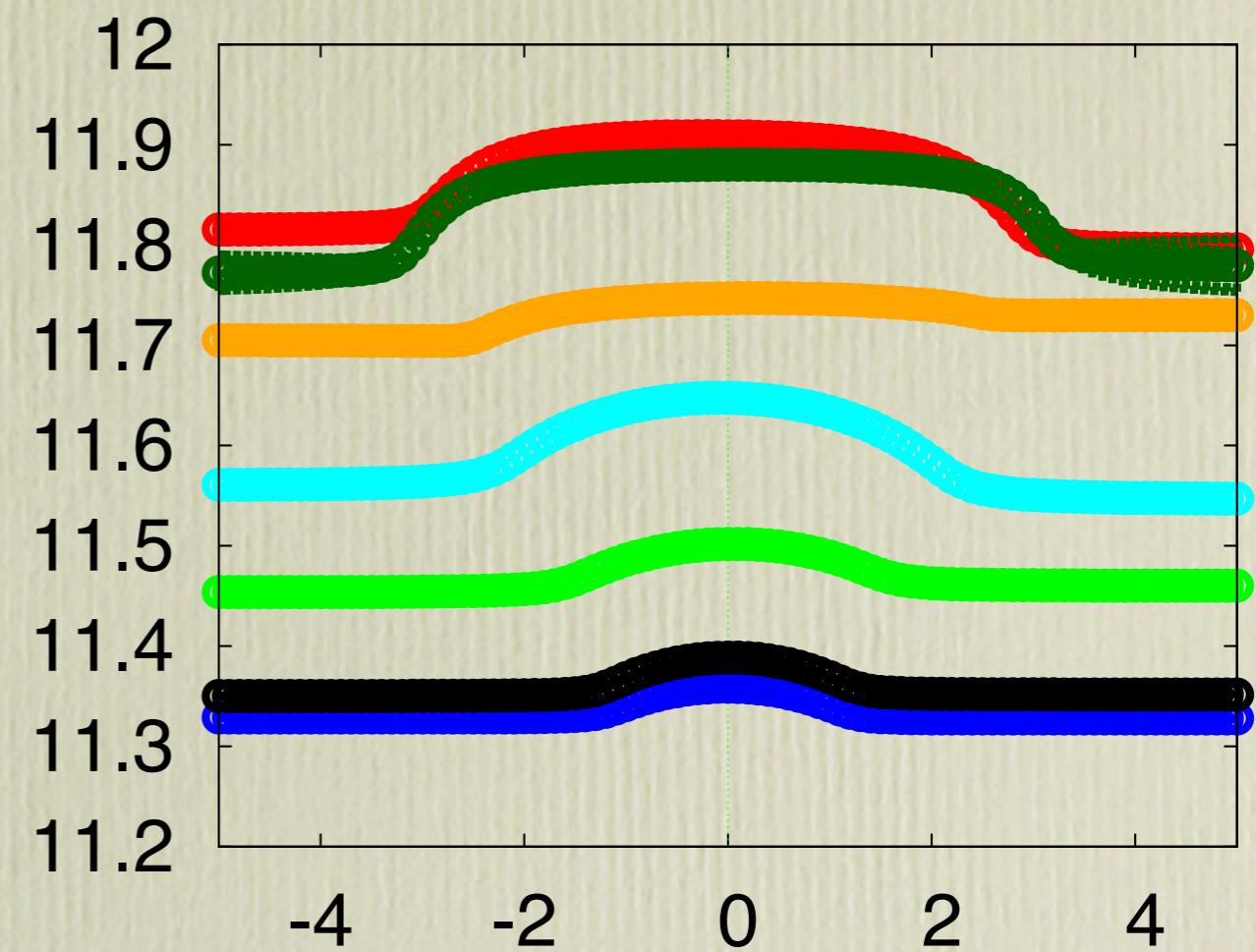
Hadronic observables

Chiral condensate in grand canonical ensemble

Lee-Yang zeros $\xi \sim 0.076$



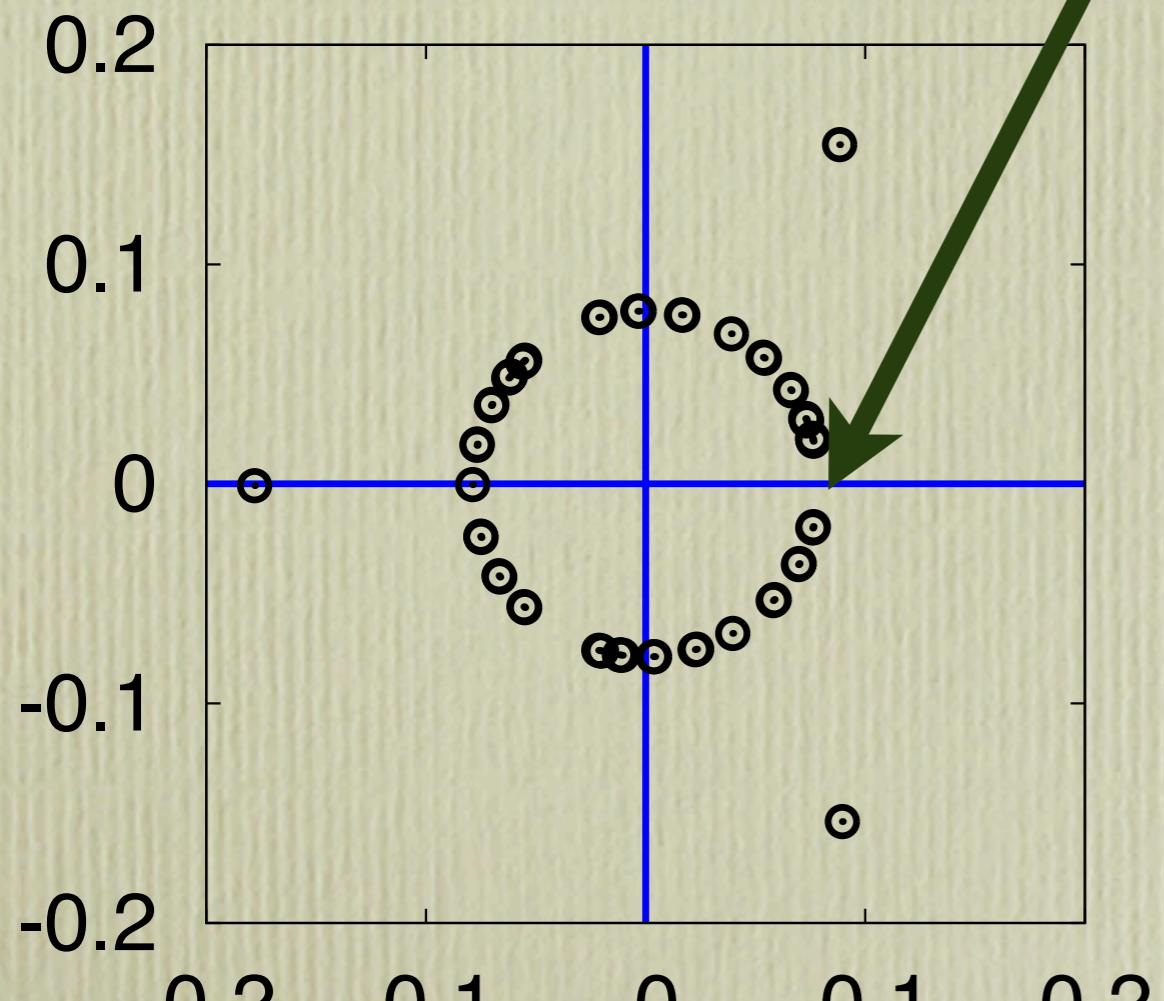
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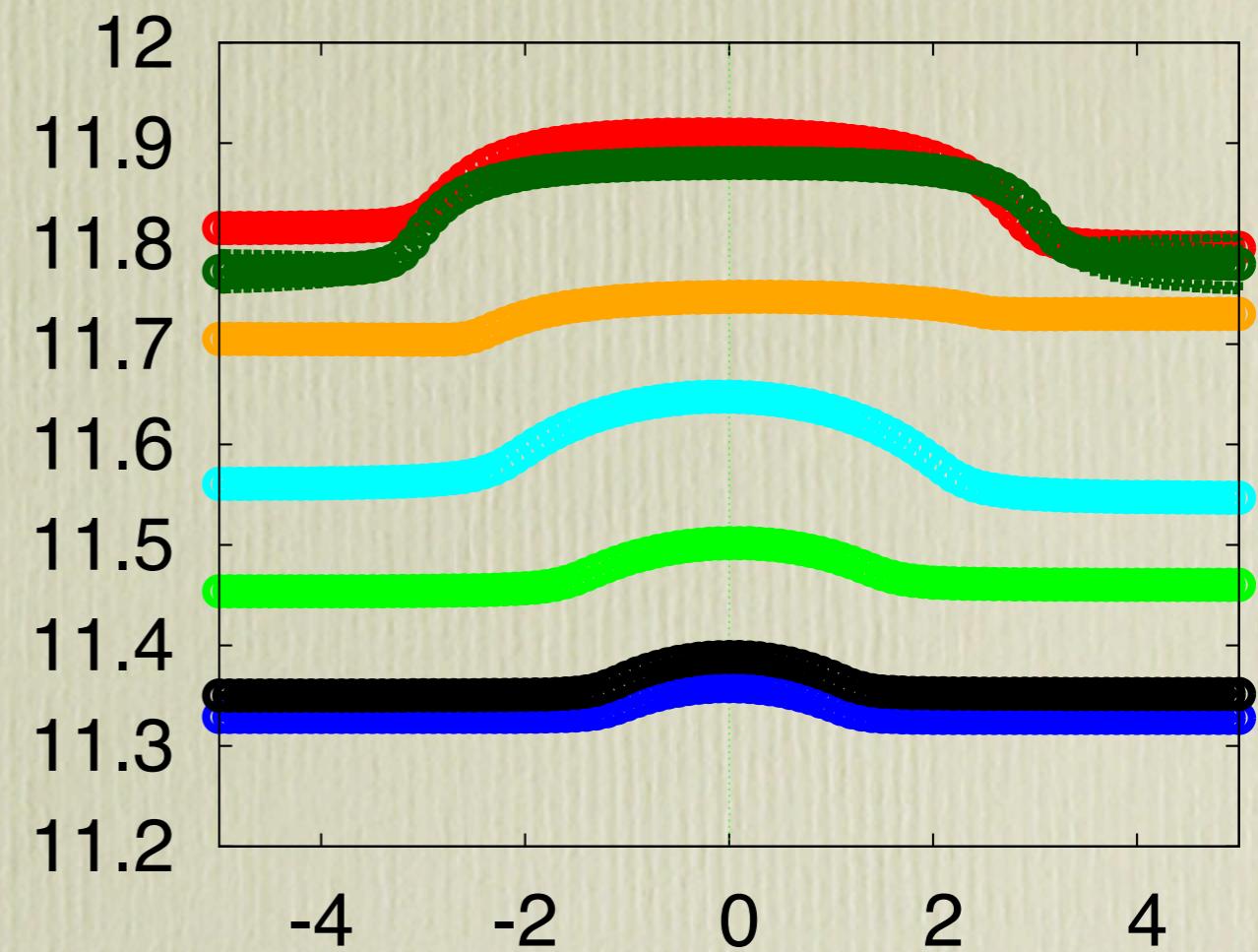
Hadronic observables

Chiral condensate in grand canonical ensemble

Lee-Yang zeros $\xi \sim 0.076 \longleftrightarrow \mu \sim -2.5$



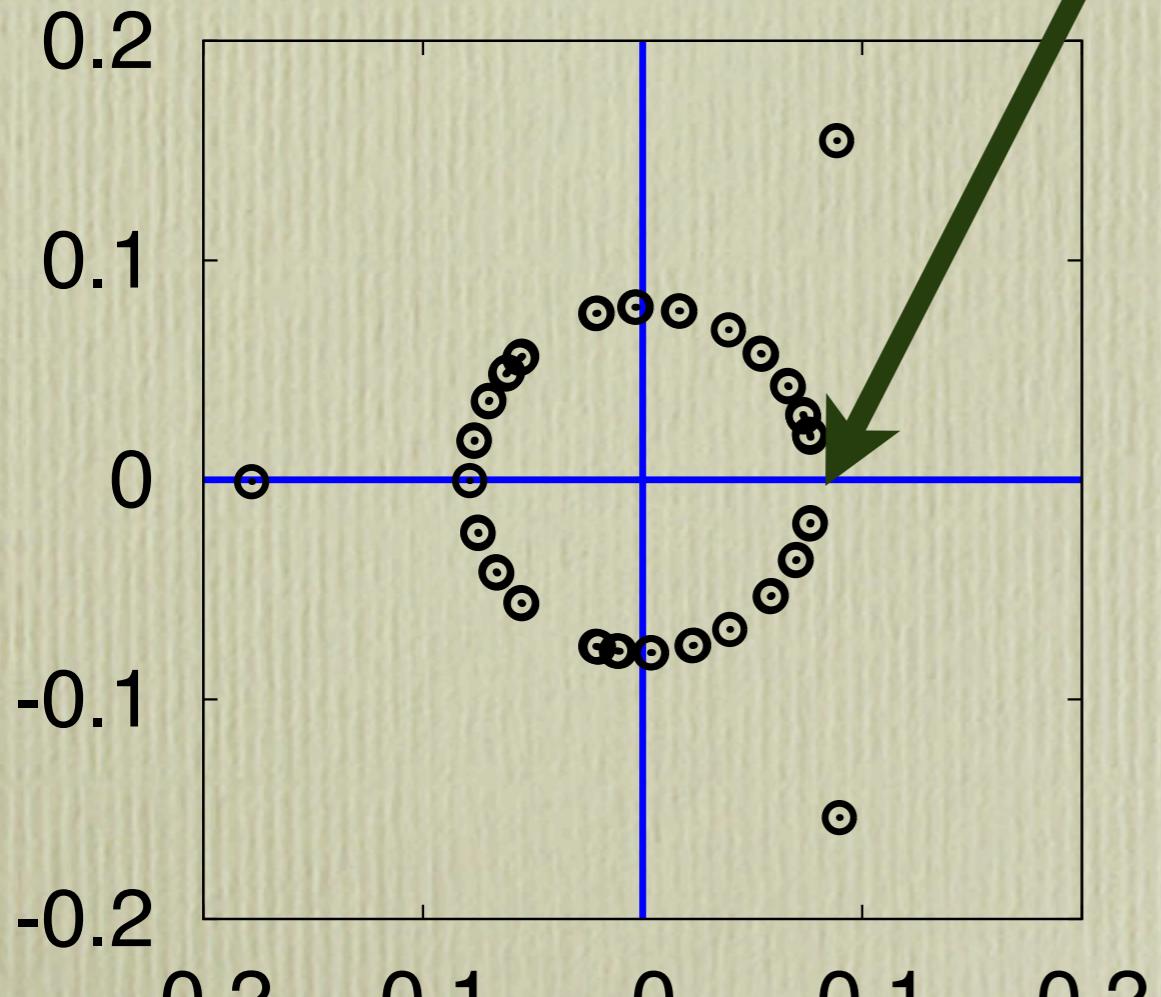
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Hadronic observables

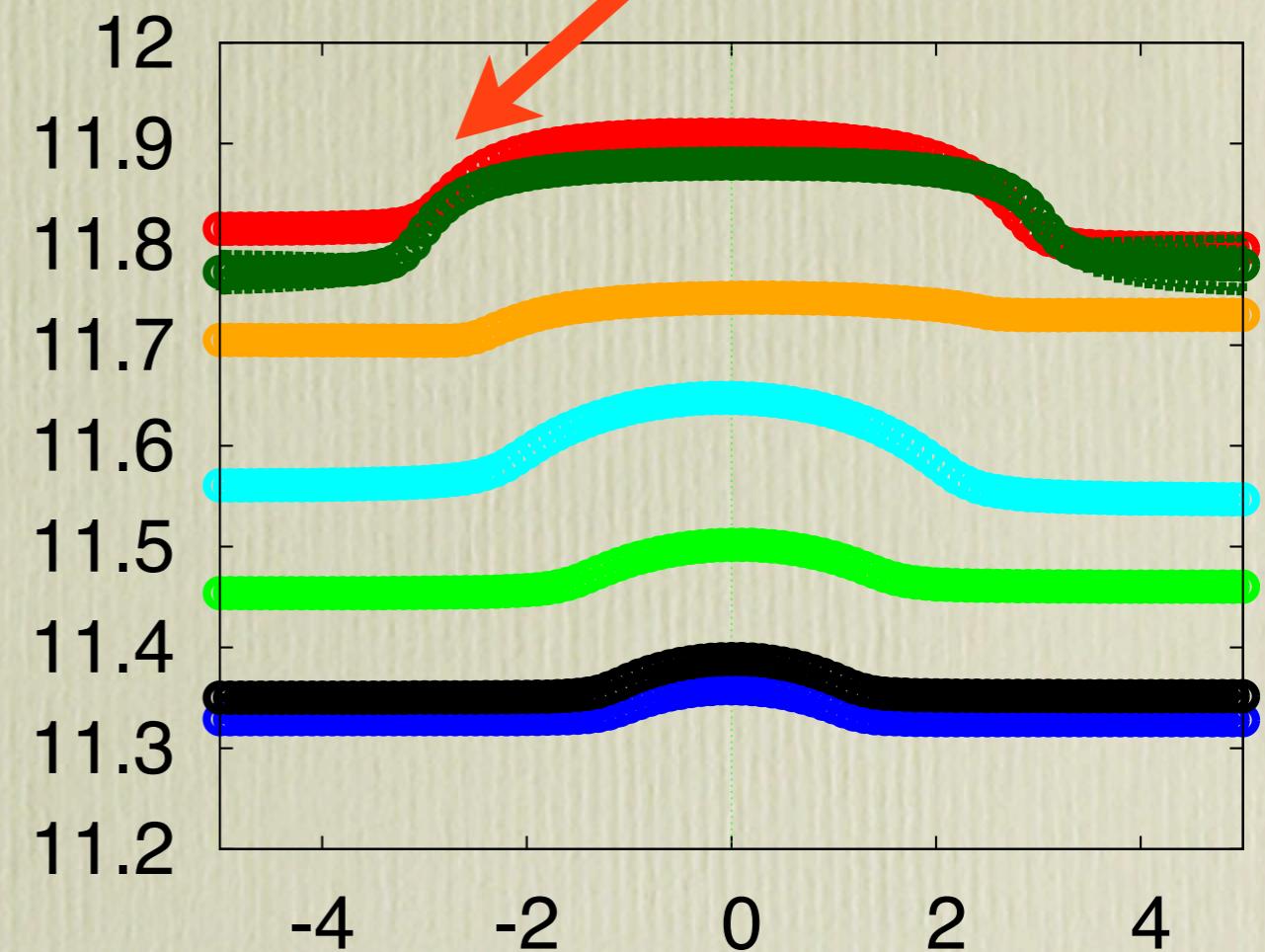
Chiral condensate in grand canonical ensemble

Lee-Yang zeros



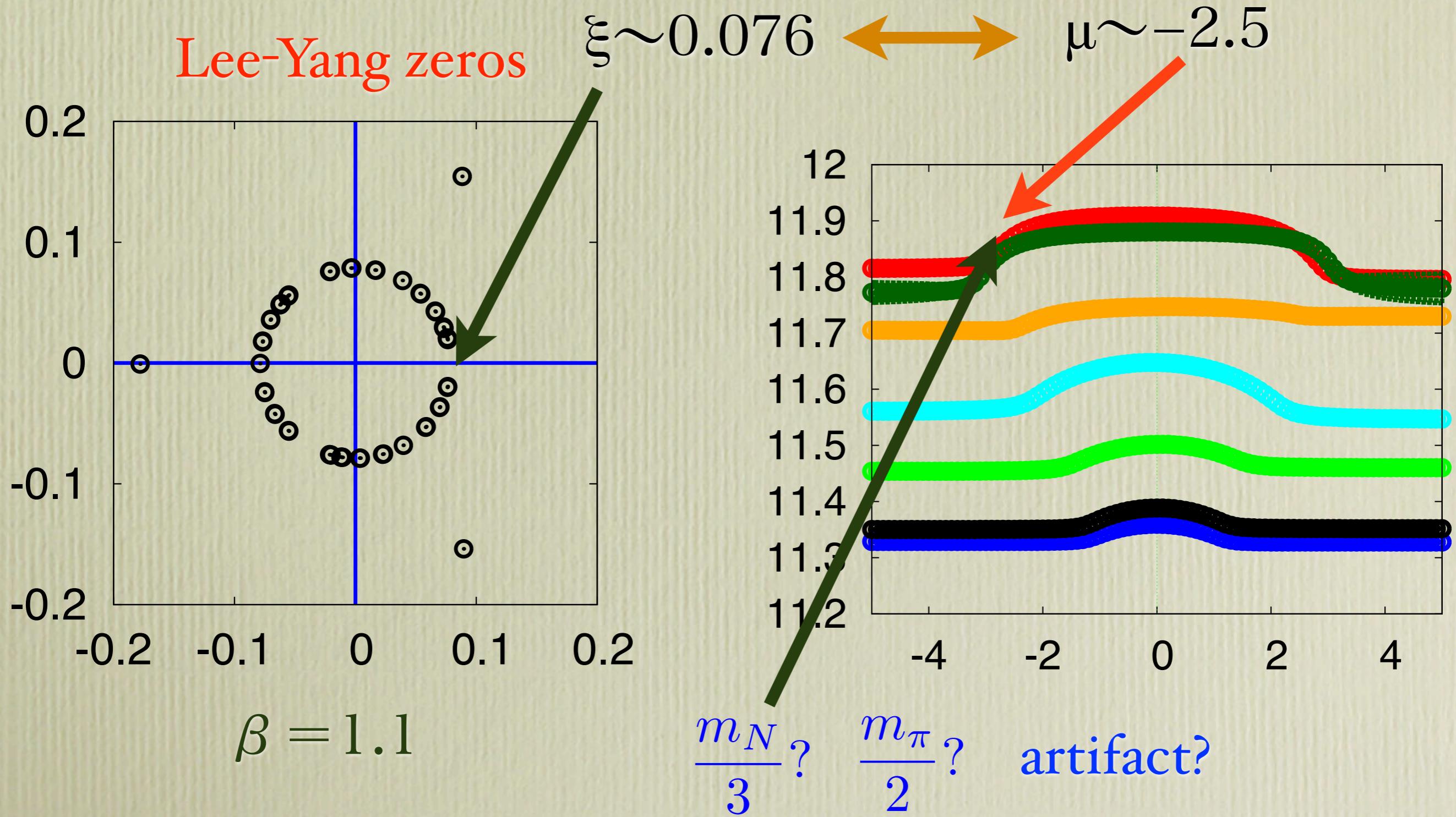
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Hadronic observables

Chiral condensate in grand canonical ensemble



Conclusion

Conclusion

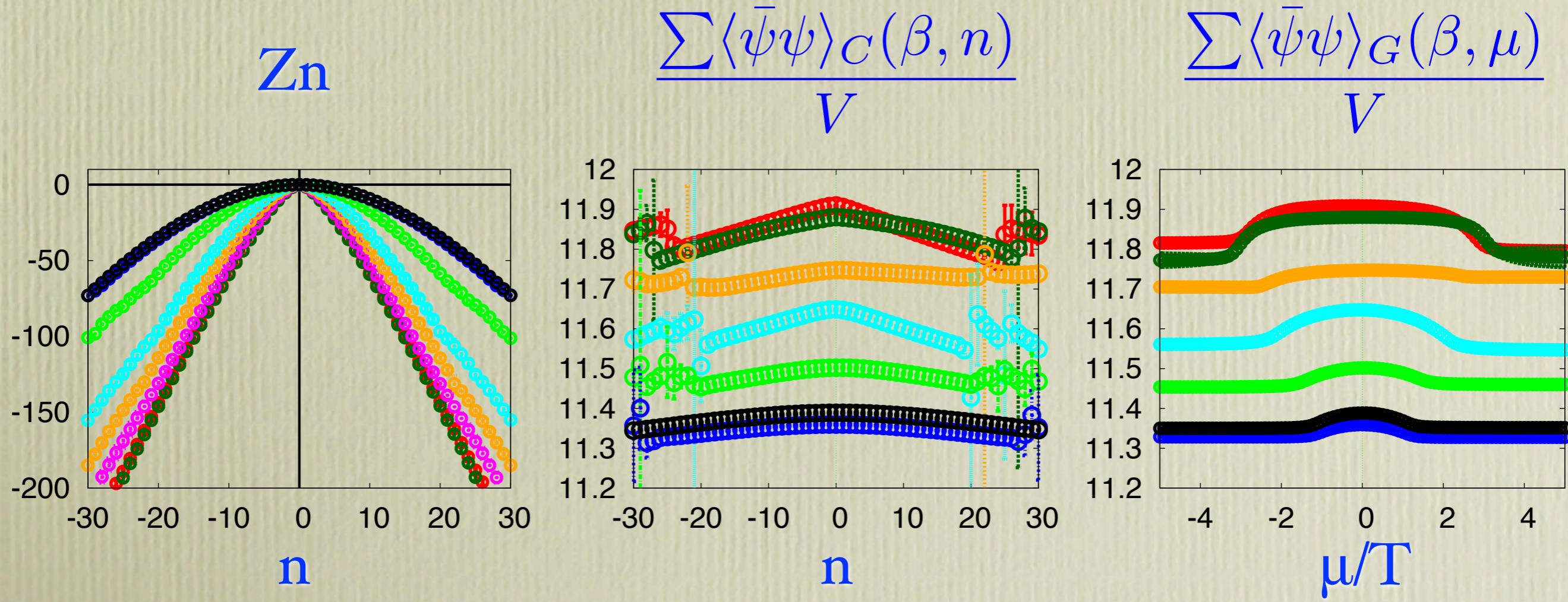
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Conclusion

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- Hopping parameter expansion works more than we expected.

Conclusion

- Canonical approach is a good choice for finite density QCD.
- Hopping parameter expansion works more than we expected.
- We have three interesting results.



If you can read this
I am on the wrong page.

Hadronic observables

Convergence radius

Hadronic observables

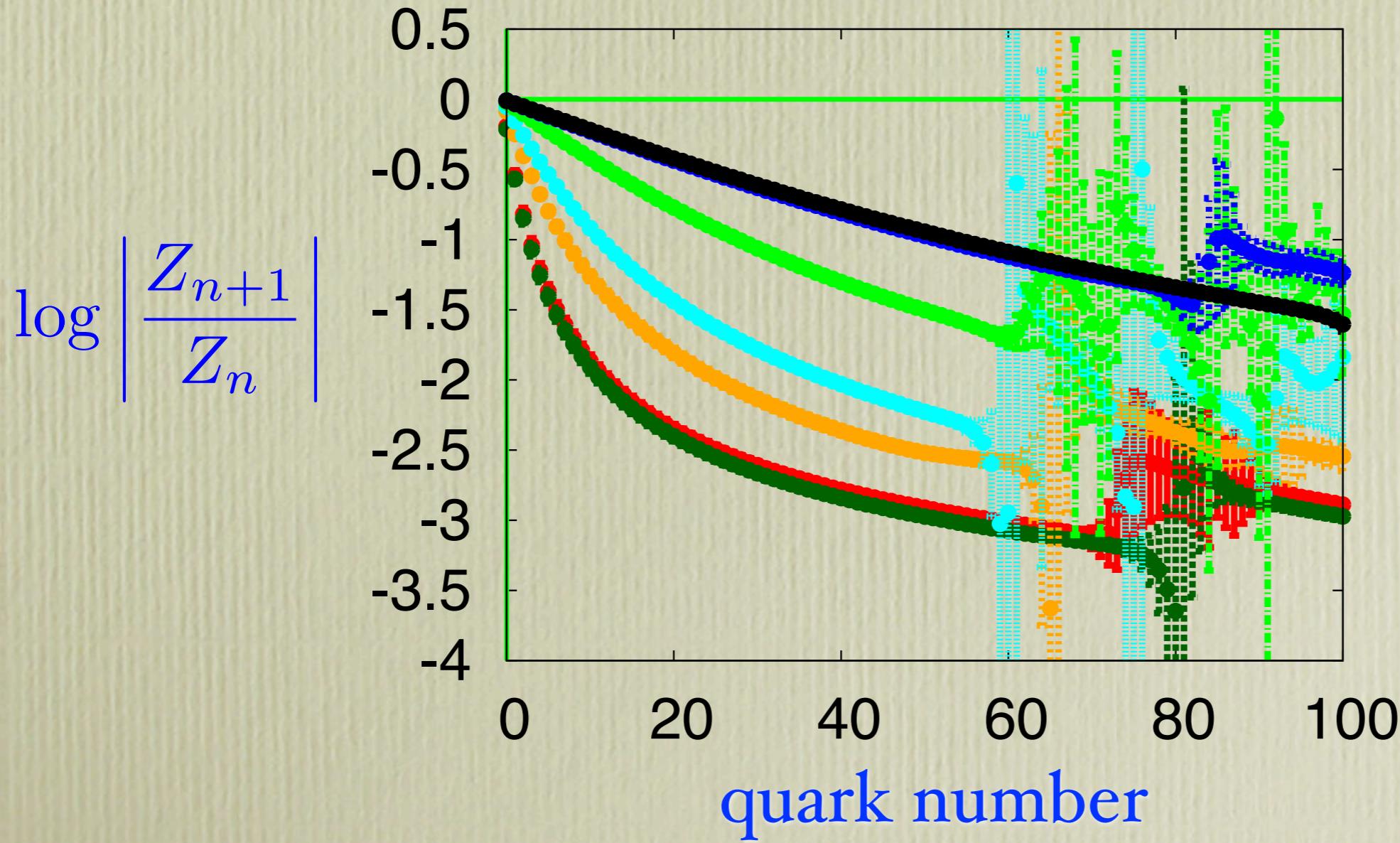
Convergence radius

$$\sum_{n=-\infty}^{\infty} Z_n \xi^n = Z_0 + Z_1 \xi + Z_2 \xi^2 + \dots + Z_{-1} \xi^{-1} + Z_{-2} \xi^{-2} + \dots$$

Hadronic observables

Convergence radius

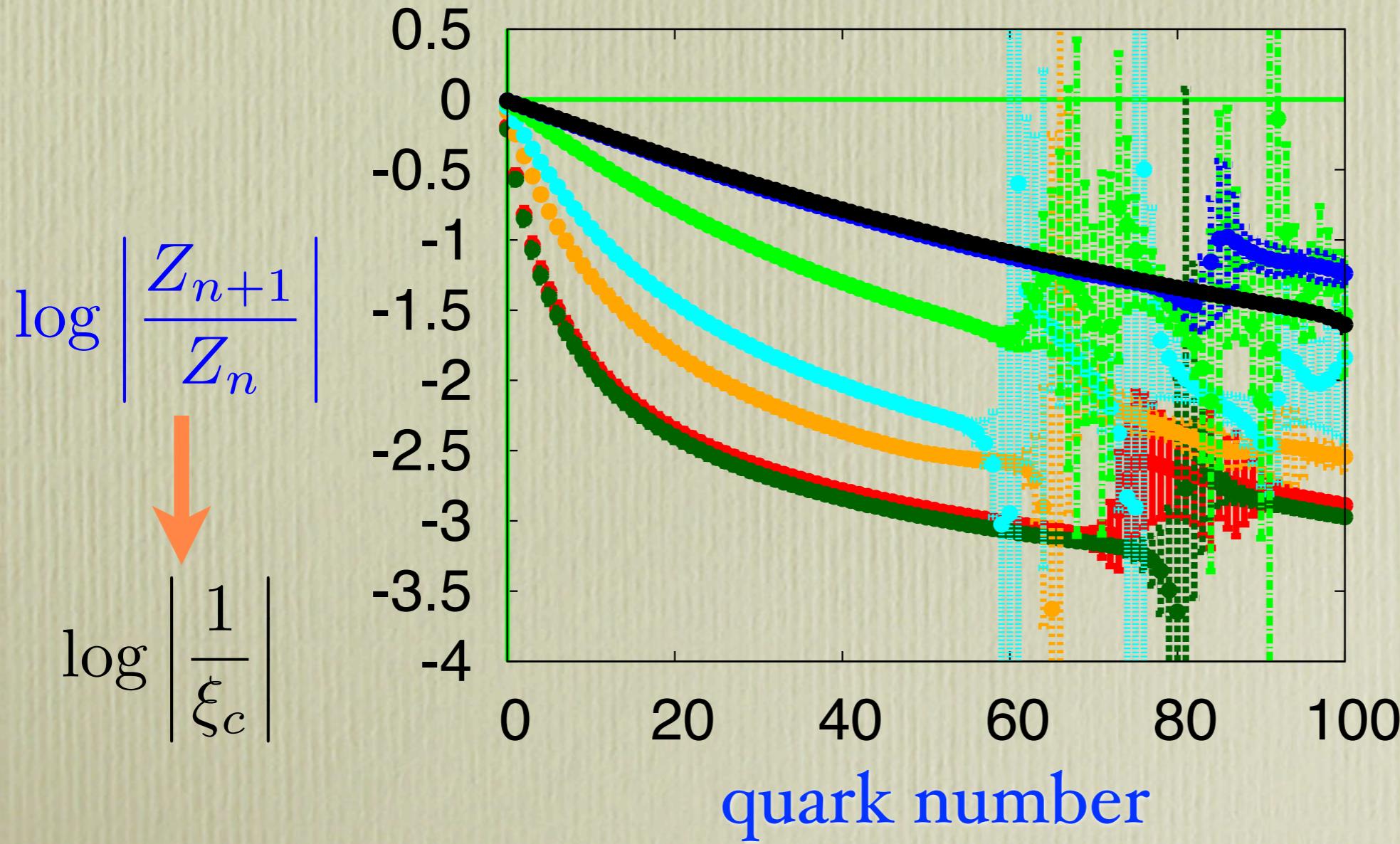
$$\sum_{n=-\infty}^{\infty} Z_n \xi^n = Z_0 + Z_1 \xi + Z_2 \xi^2 + \dots + Z_{-1} \xi^{-1} + Z_{-2} \xi^{-2} + \dots$$



Hadronic observables

Convergence radius

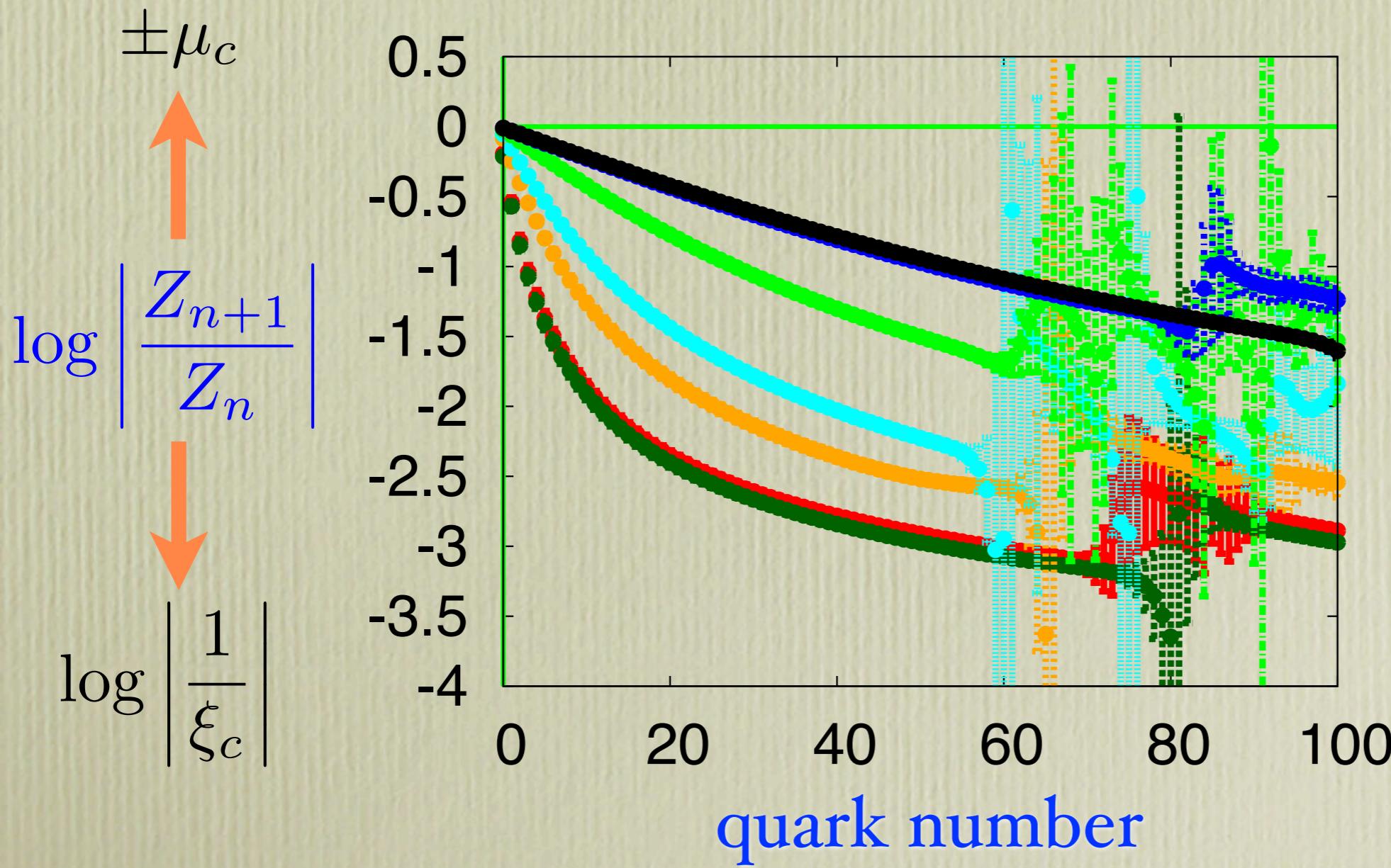
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Hadronic observables

Convergence radius

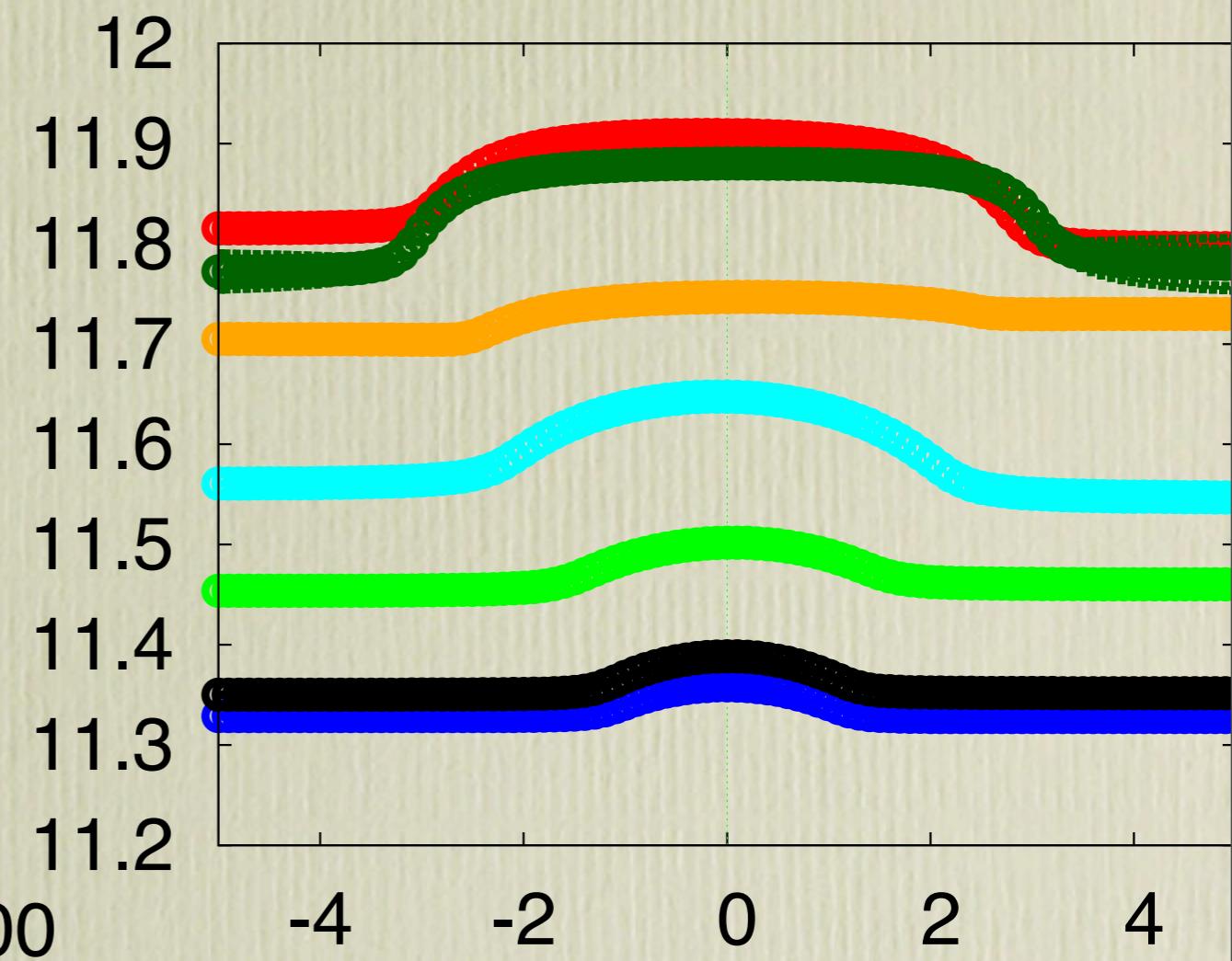
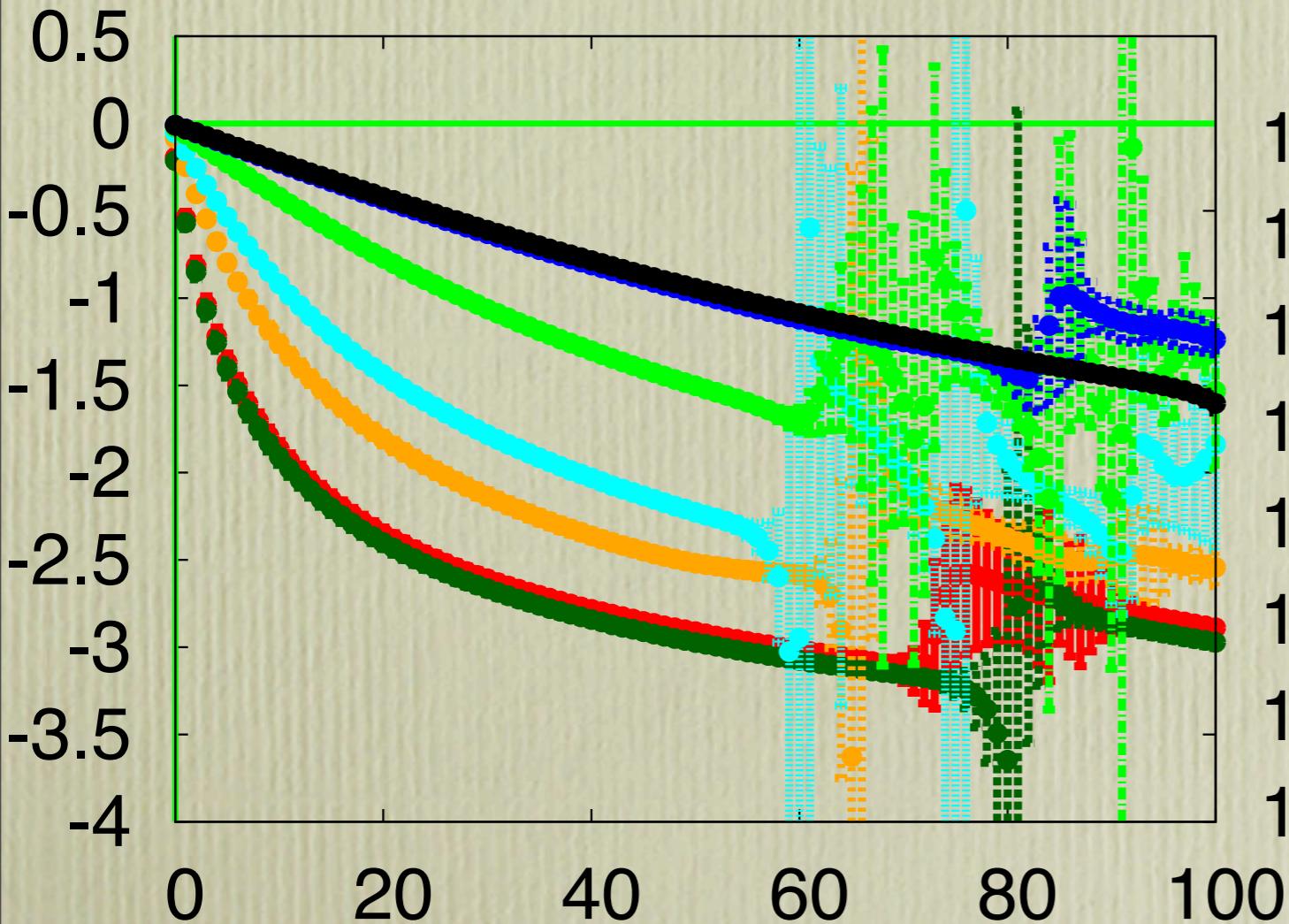
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Convergence radius

$$\sum_{n=-\infty}^{\infty} Z_n \xi^n = Z_0 + Z_1 \xi + Z_2 \xi^2 + \dots + Z_{-1} \xi^{-1} + Z_{-2} \xi^{-2} + \dots$$



Numerical results Phase(Zc(n))

Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

Numerical results Phase($Z_C(n)$)

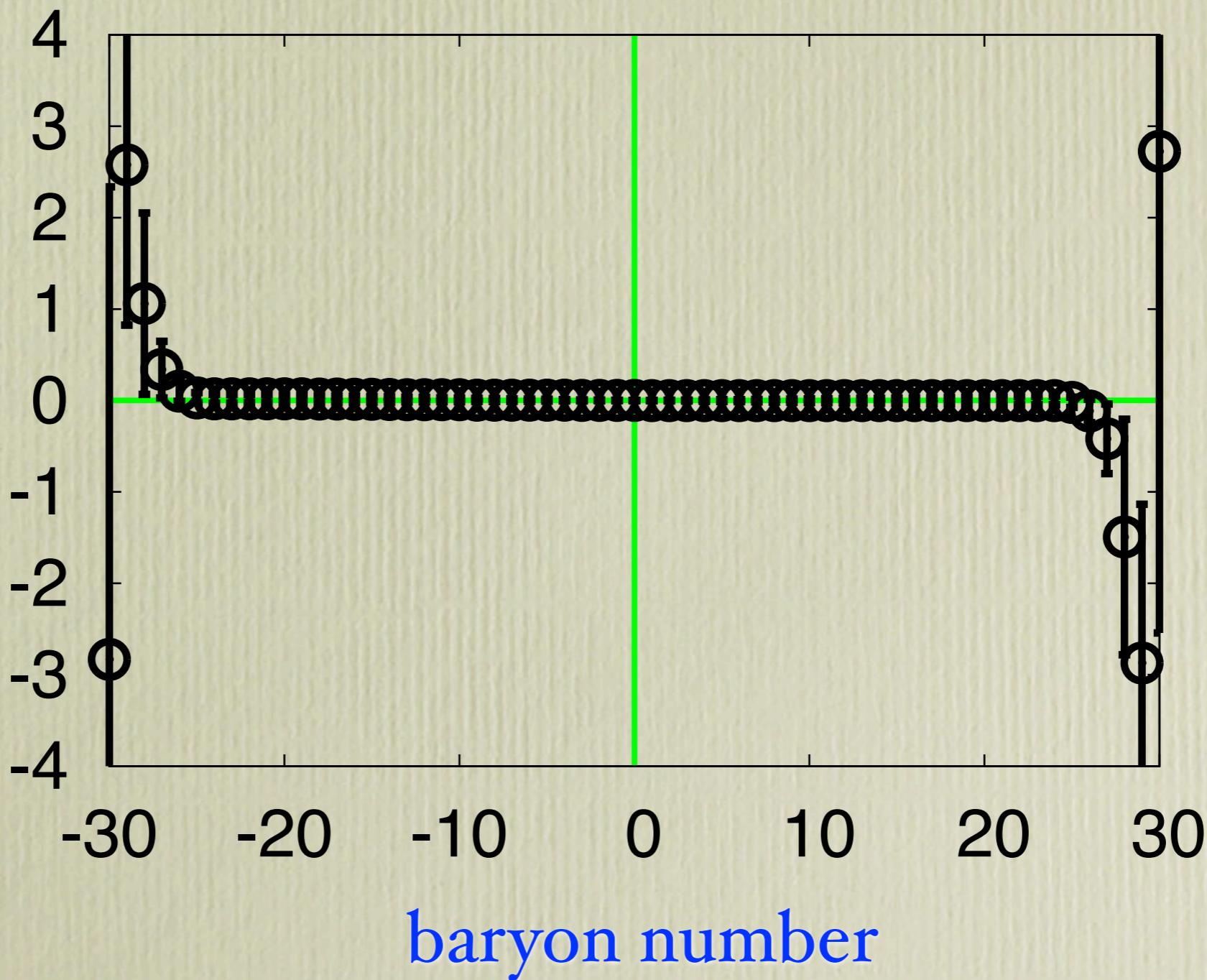
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$\beta = 1.9$

$\mu = 0$

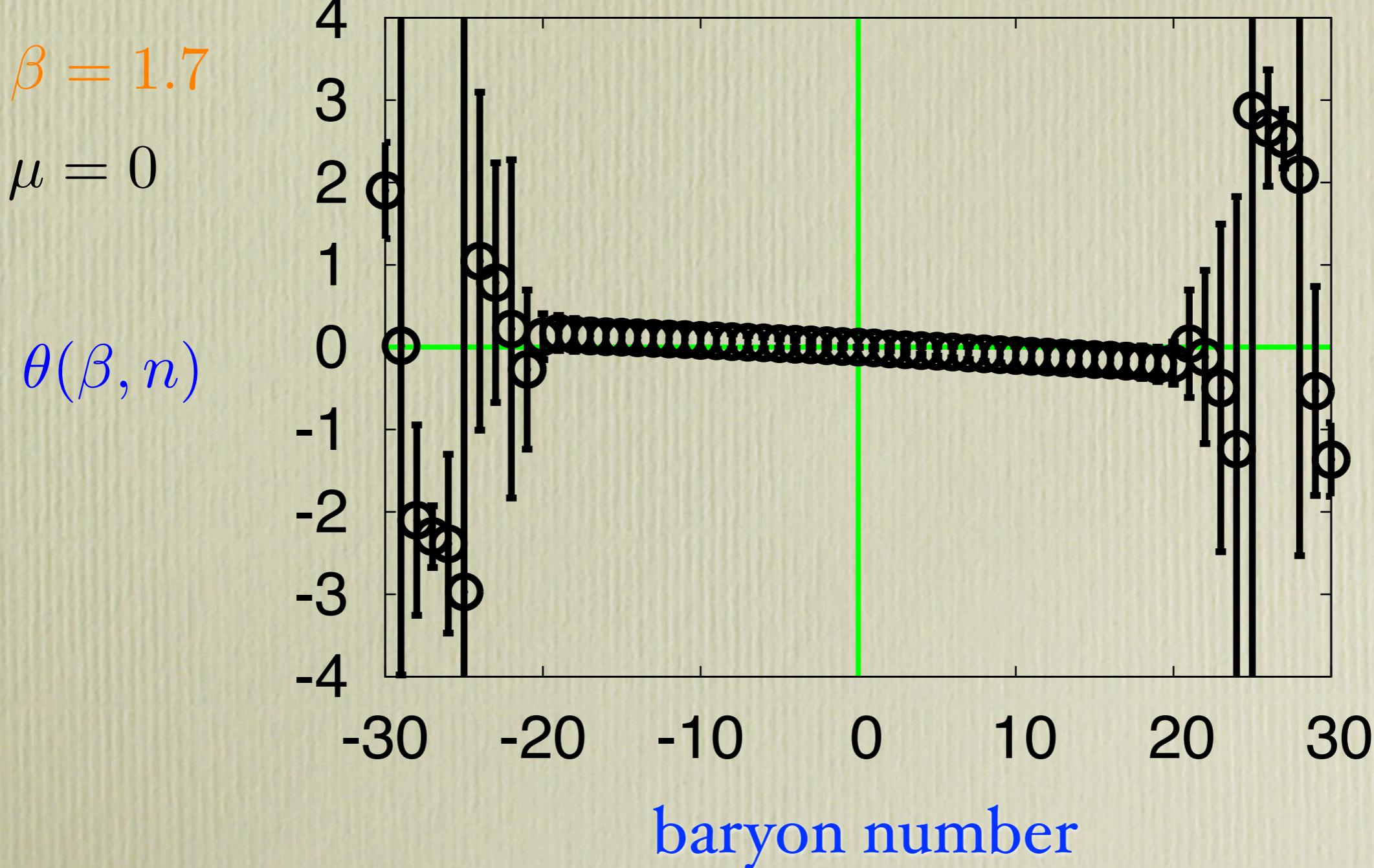
$\theta(\beta, n)$



Numerical results Phase($Z_c(n)$)

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Numerical results Phase($Z_C(n)$)

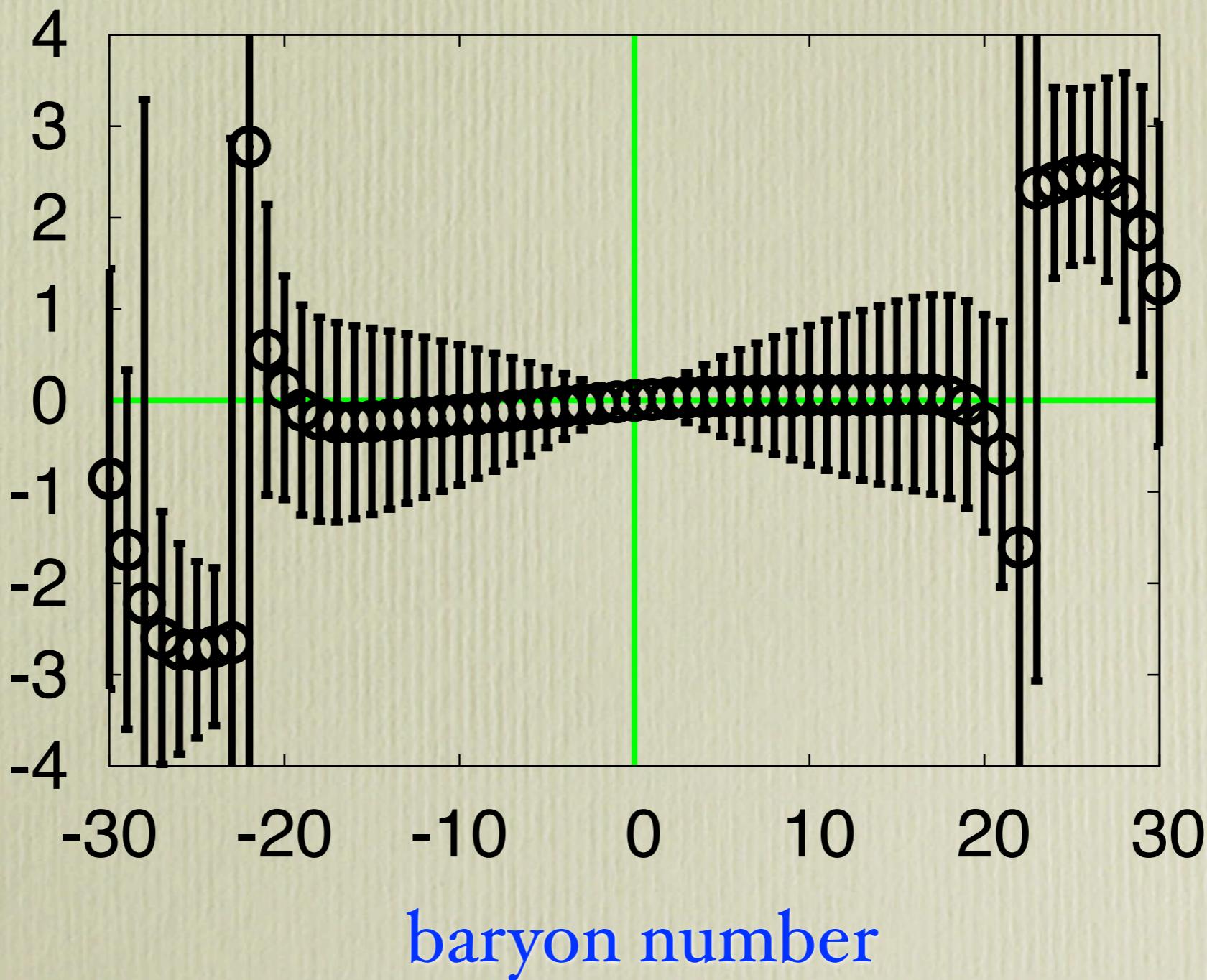
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$\beta = 1.5$

$\mu = 0$

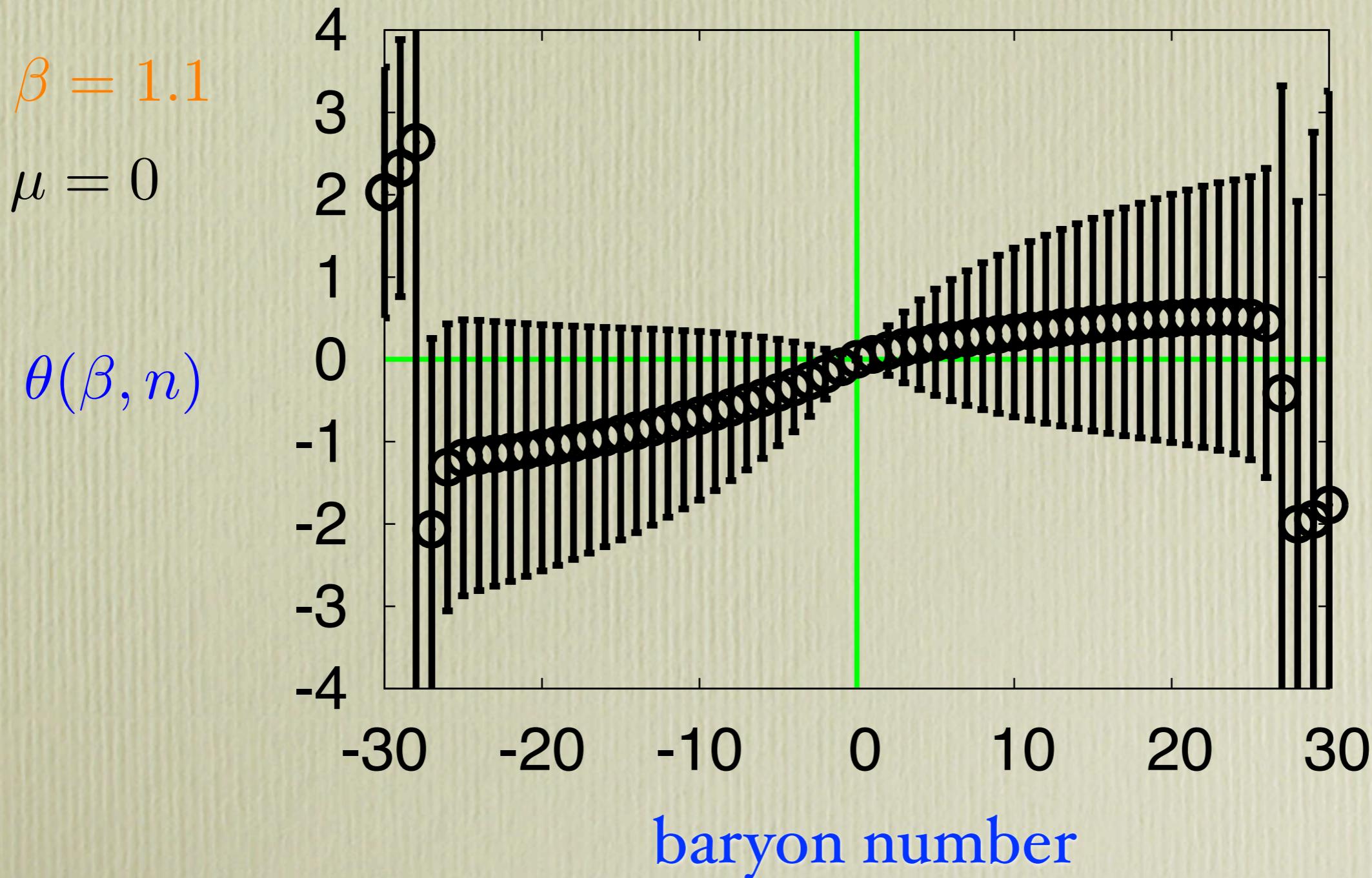
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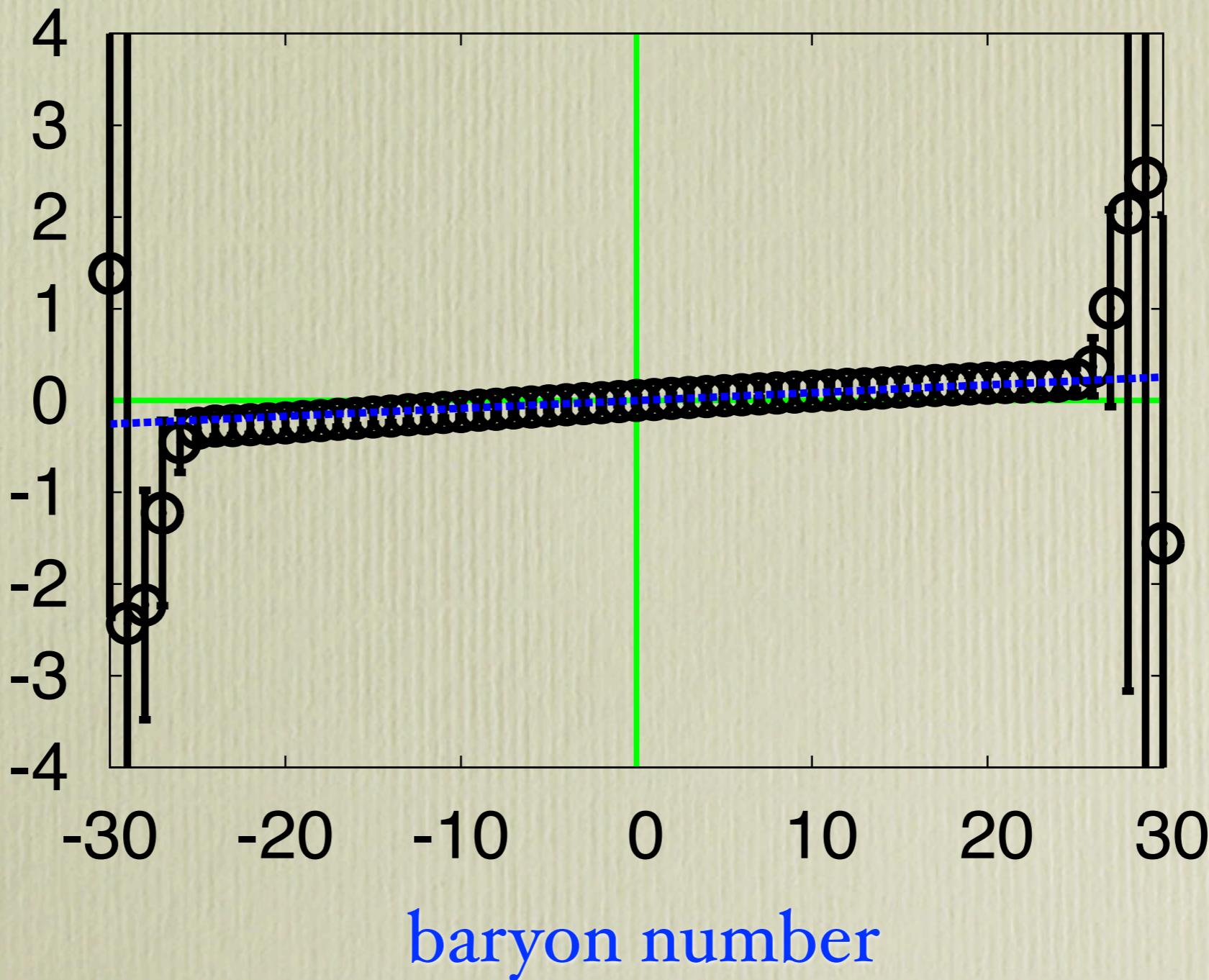
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$\beta = 1.9$

$\mu = 0.125$

$\theta(\beta, n)$



Numerical results Phase($Z_C(n)$)

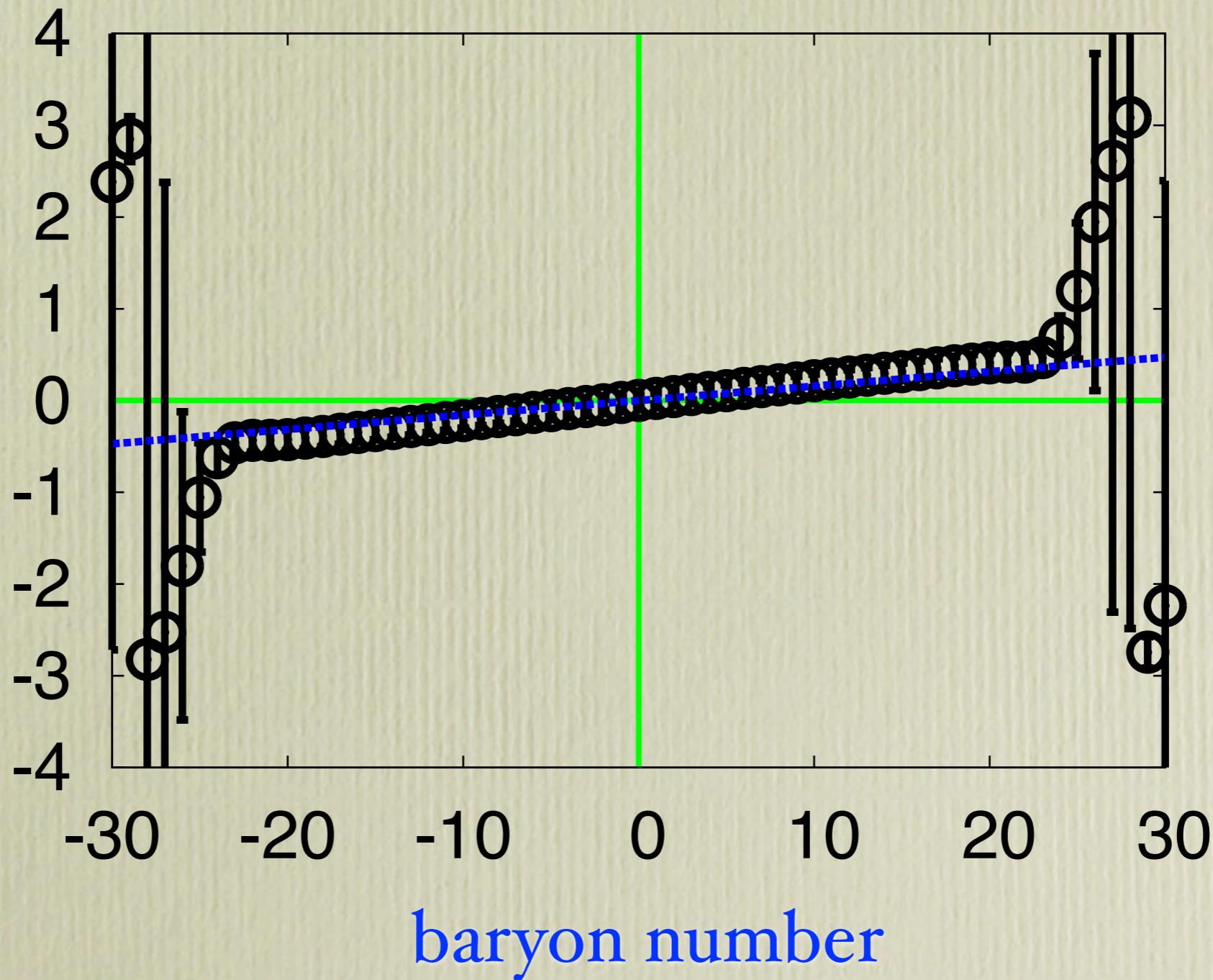
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$\beta = 1.9$

$\mu = 0.25$

$\theta(\beta, n)$



Numerical results Phase($Z_C(n)$)

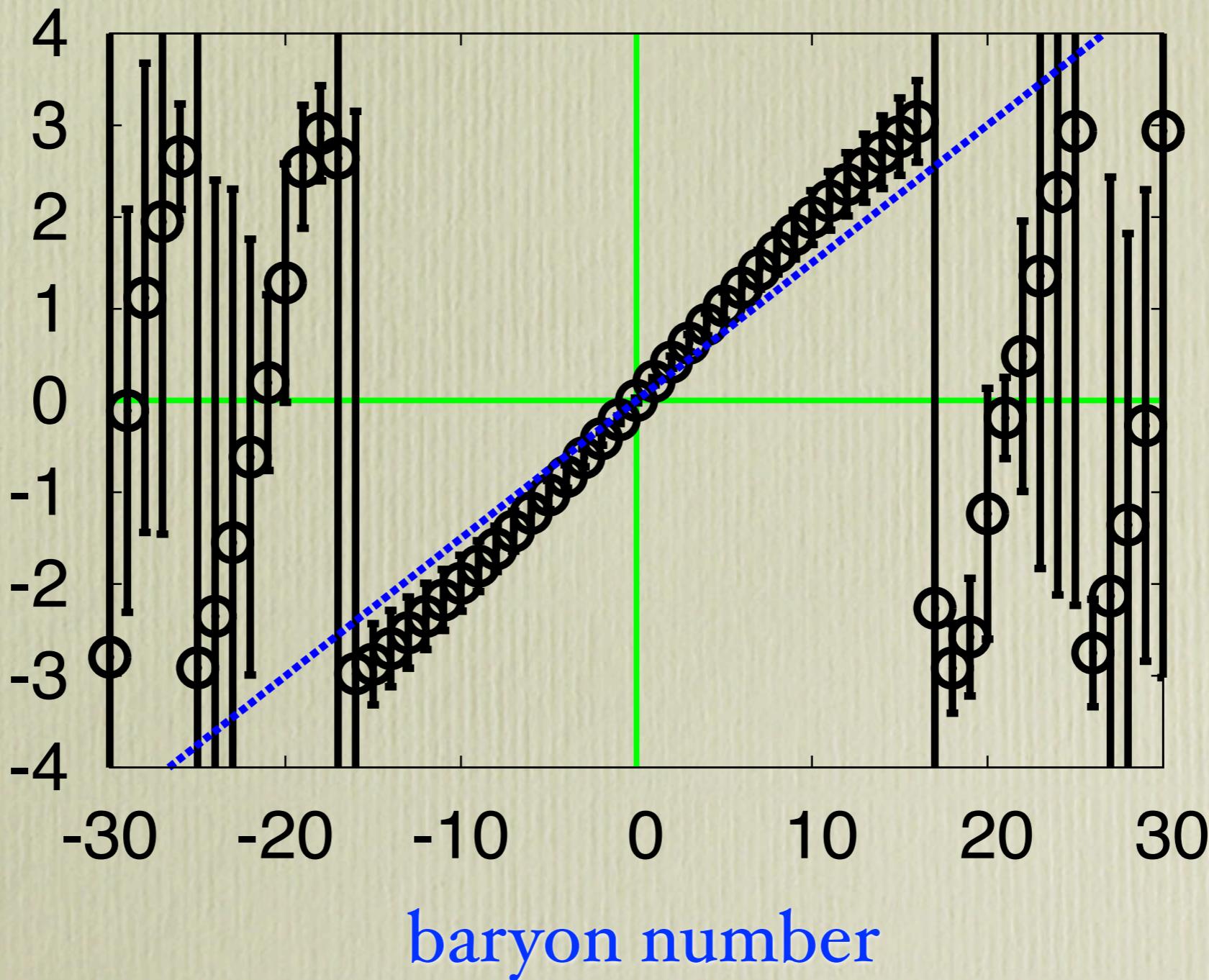
Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\beta = 1.7$$

$$\mu = 0.125$$

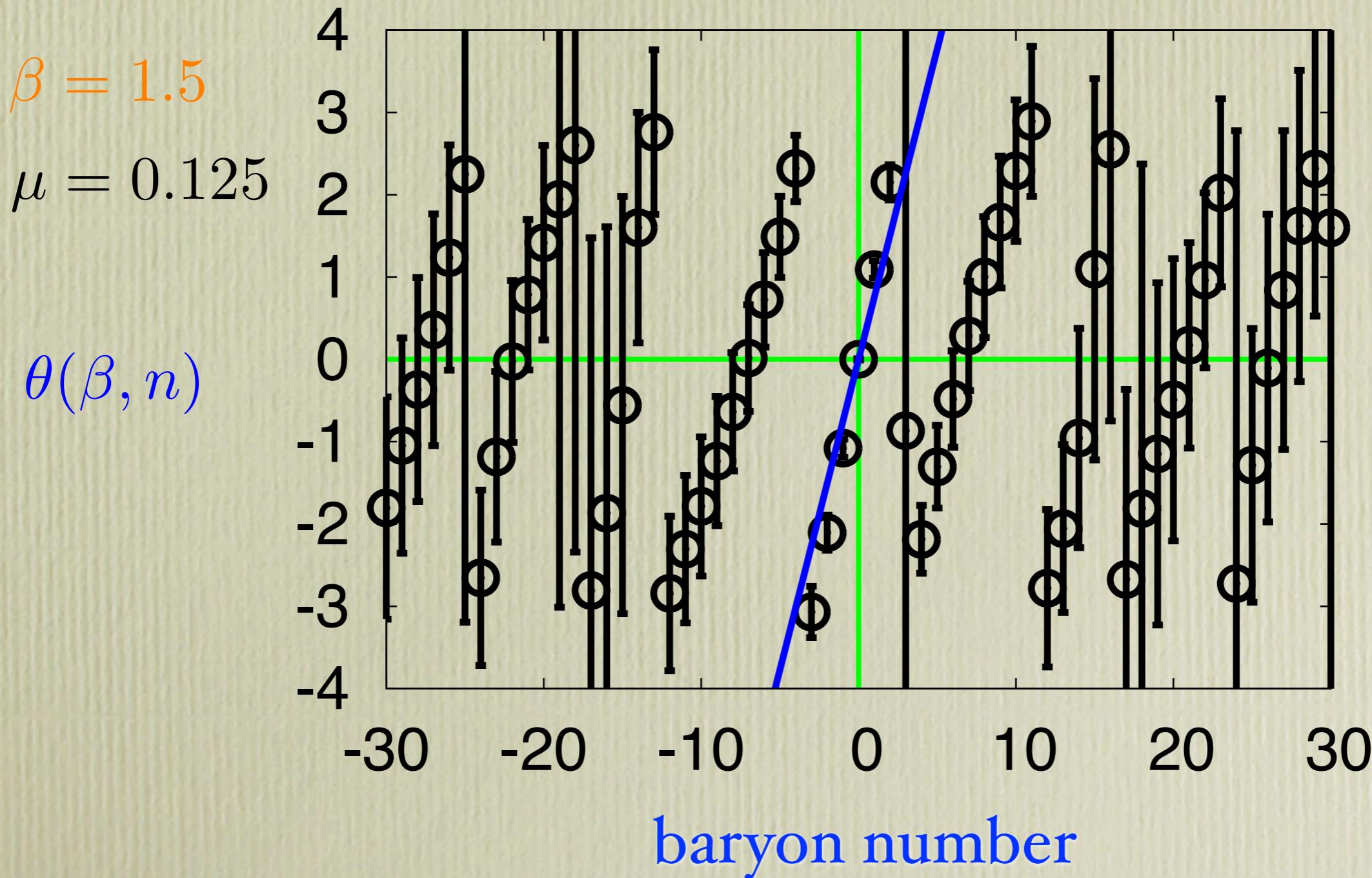
$$\theta(\beta, n)$$



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Canonical partition function

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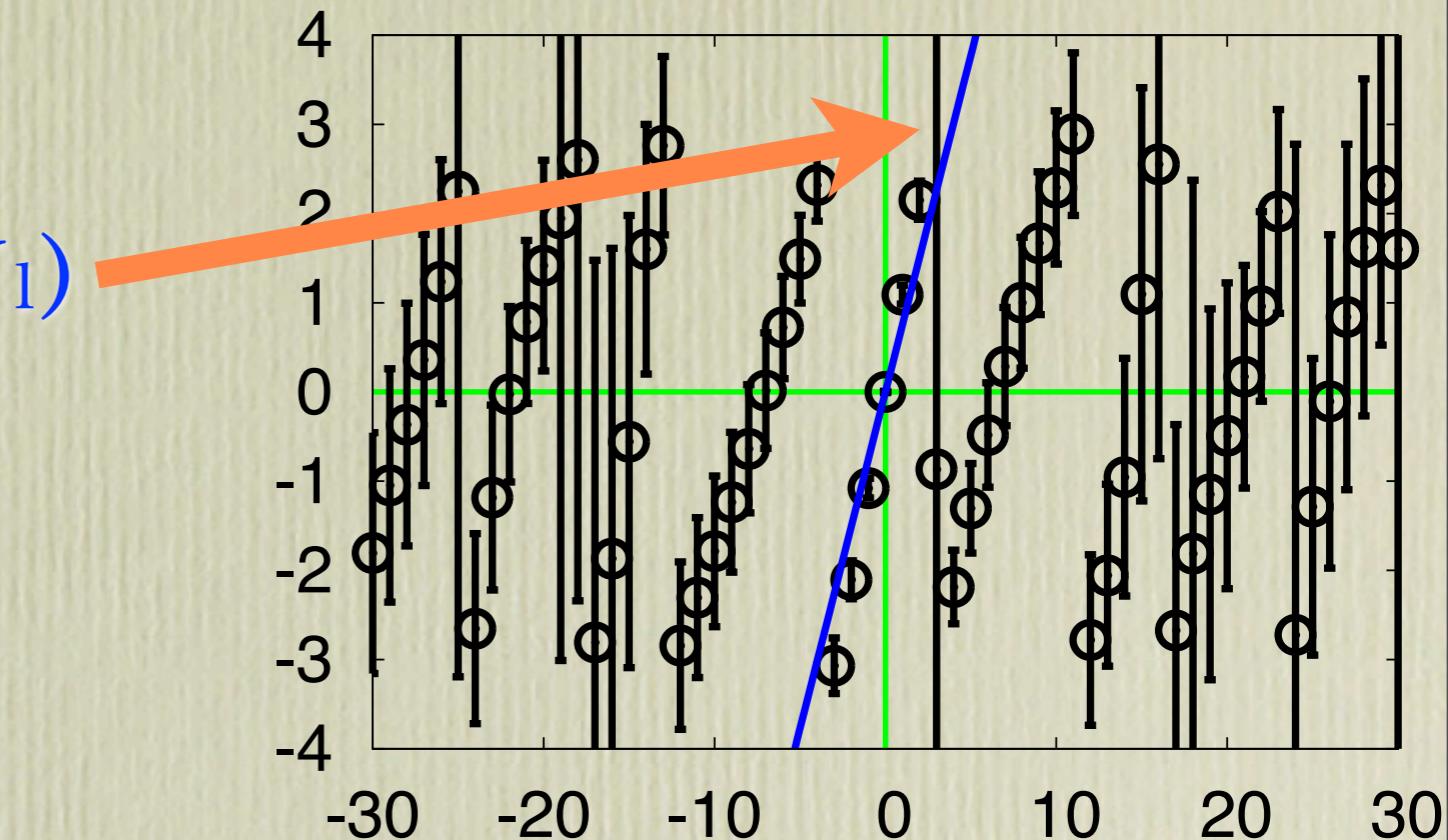


Numerical results Phase($Z_C(n)$)

Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

slope $\sim 2 \times \text{phase}(W_1)$



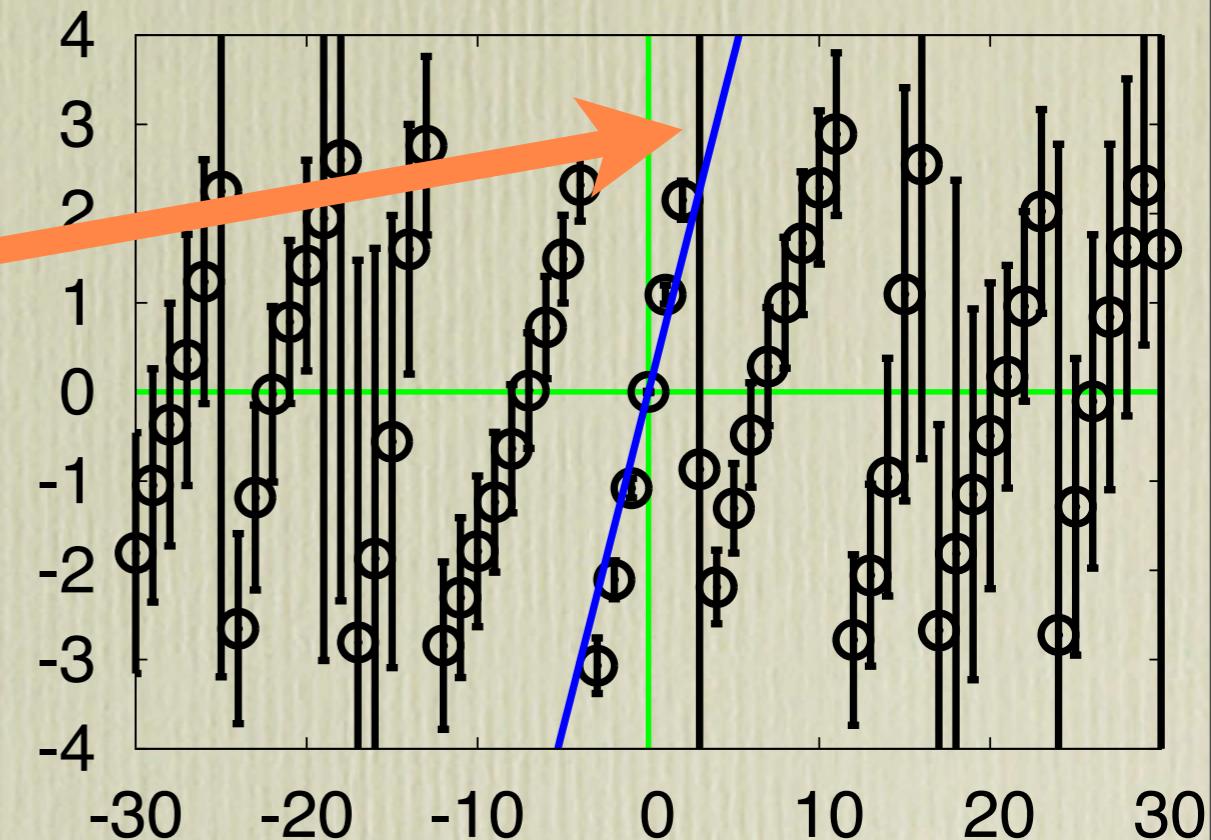
Numerical results Phase($Z_C(n)$)

Canonical partition function

$$Z_C(T, n, V) = |Z_C(\beta, n)| e^{i\theta(\beta, n)}$$

$$\text{TrLog} D_W(\mu) = \sum_{N=-\infty}^{\infty} W_N \xi^N$$

slope $\sim 2 \times \text{phase}(W_1)$



Numerical results Phase($Z_C(n)$)

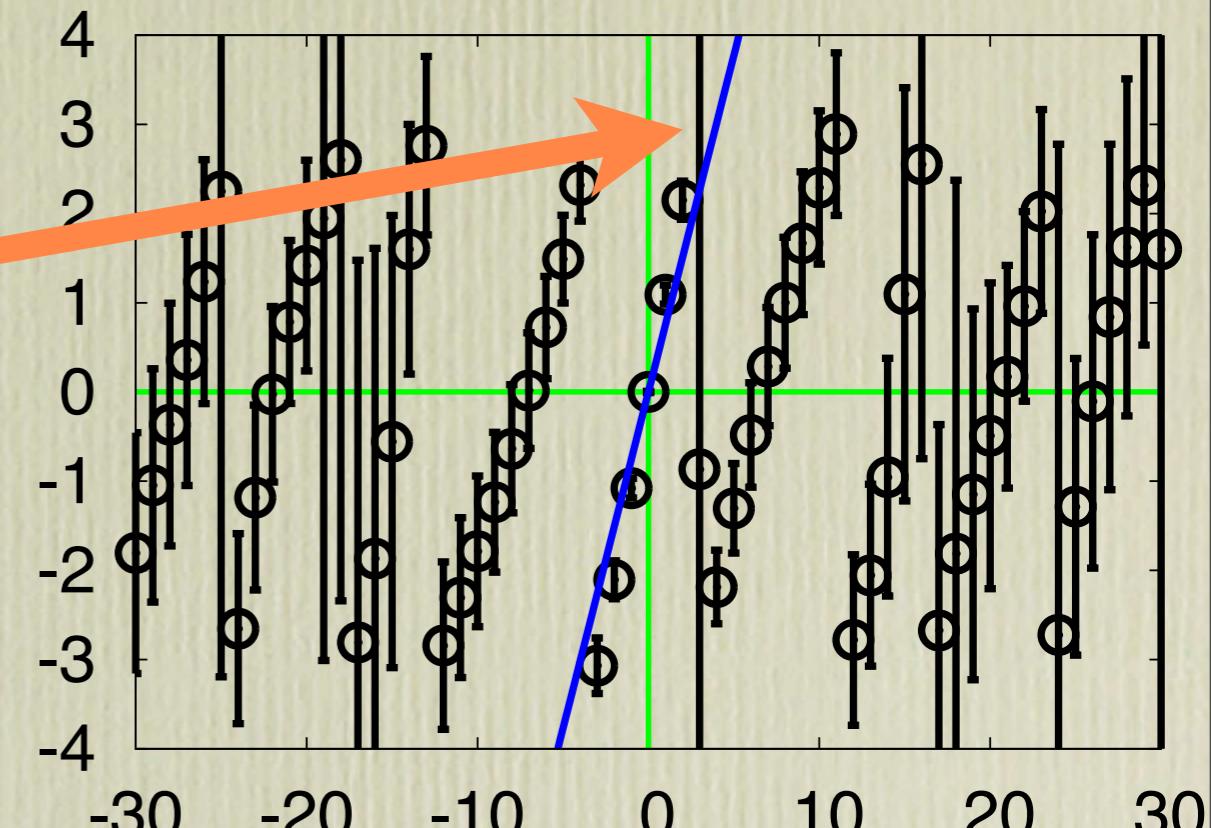
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phase of $\text{Det} D_W(\mu)$



Numerical results Phase($Z_C(n)$)

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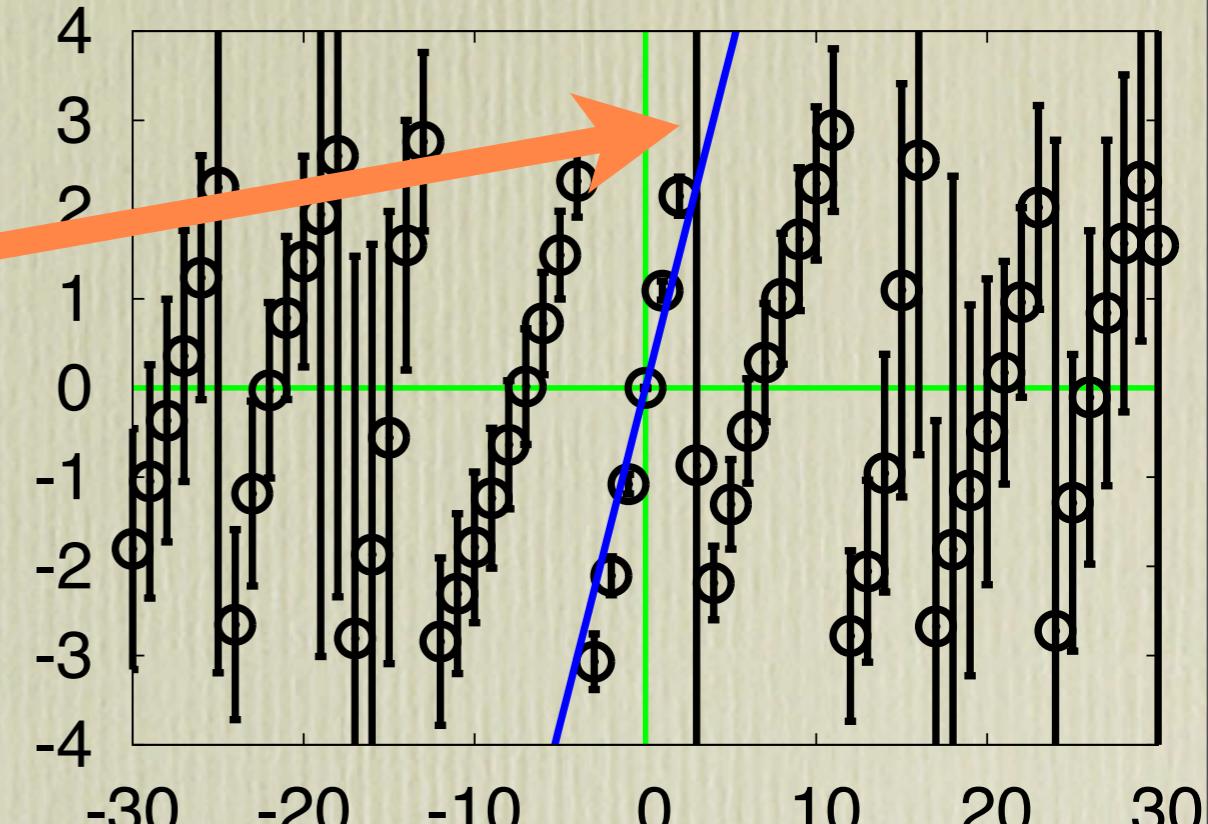
slope $\sim 2 \times \text{phase}(W_1)$

phase of $\text{Det} D_W(\mu)$

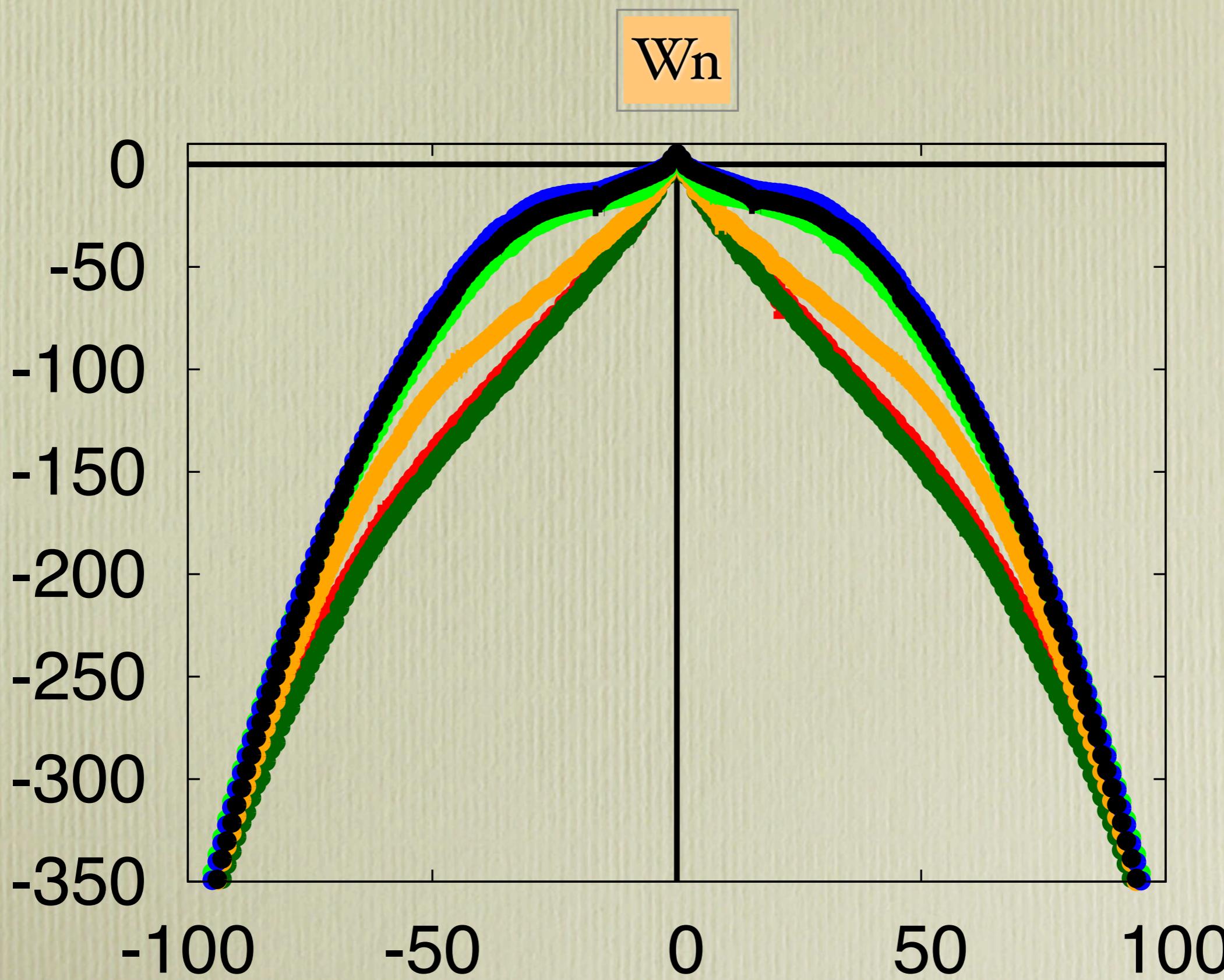


Pion condensate

Ipsen and Splittorff (2012)



Hadronic observables

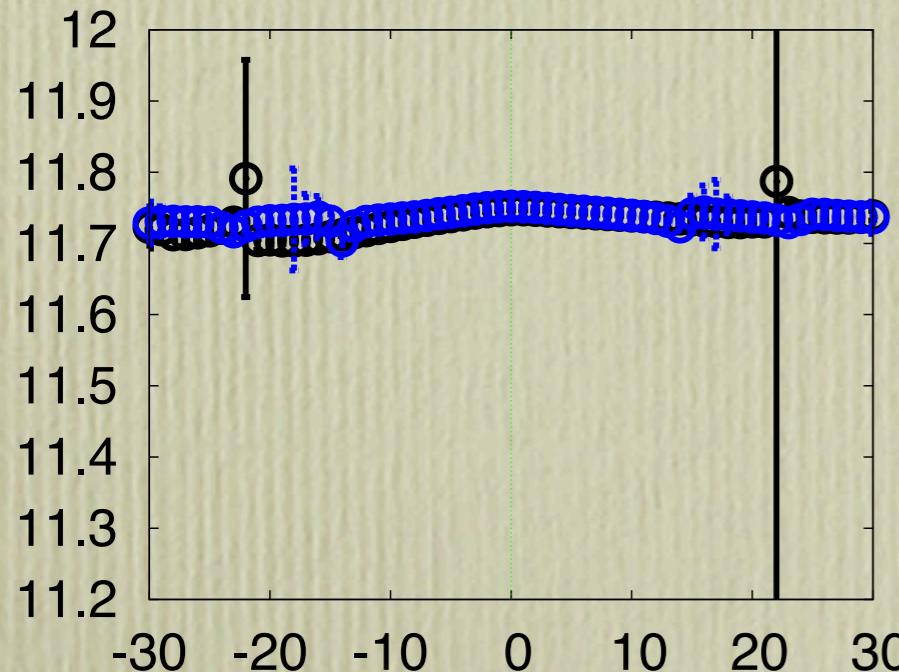


Hadronic observables

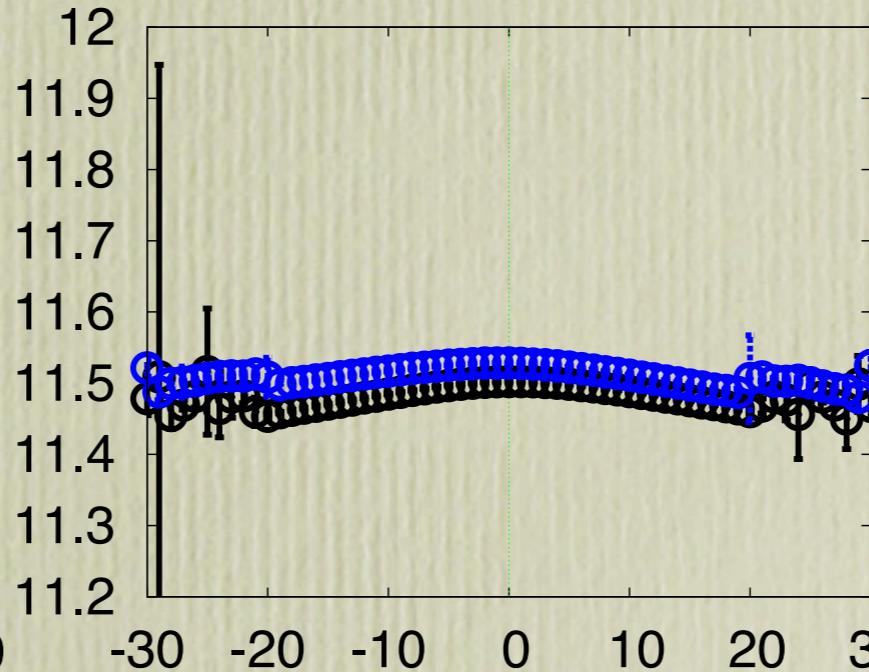
Chiral condensate in canonical ensemble

$$\frac{\sum \langle \bar{\psi} \psi \rangle_C(\beta, n)}{V}$$

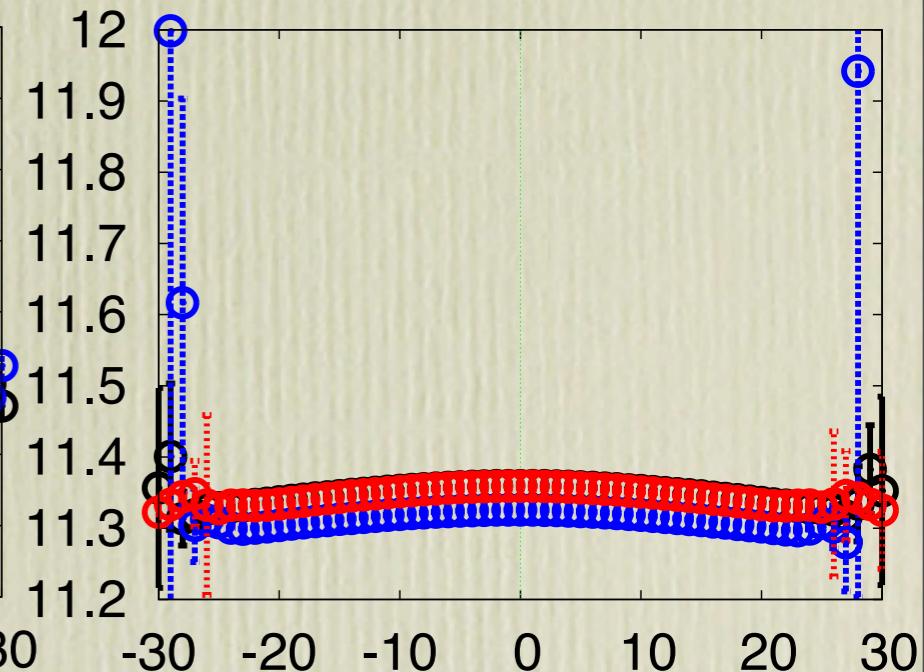
μ dependence



$\beta = 1.5$



$\beta = 1.7$



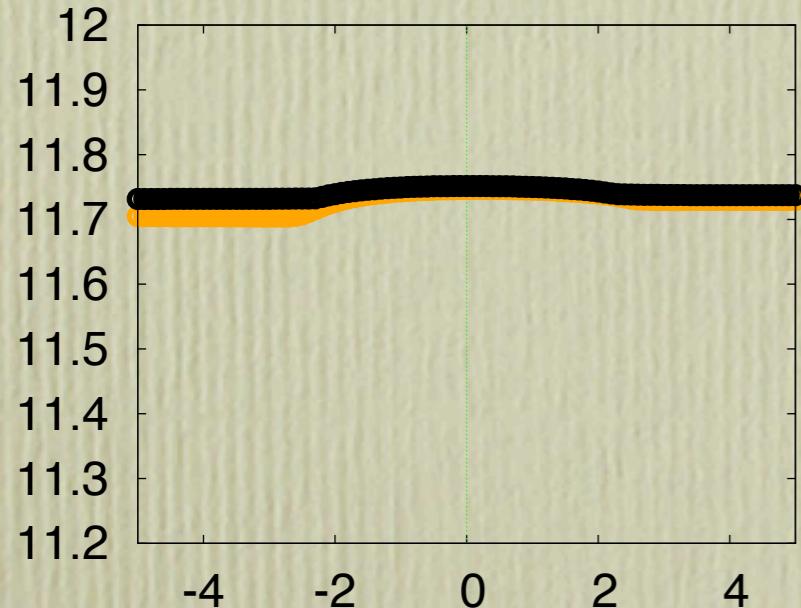
$\beta = 1.9$

Hadronic observables

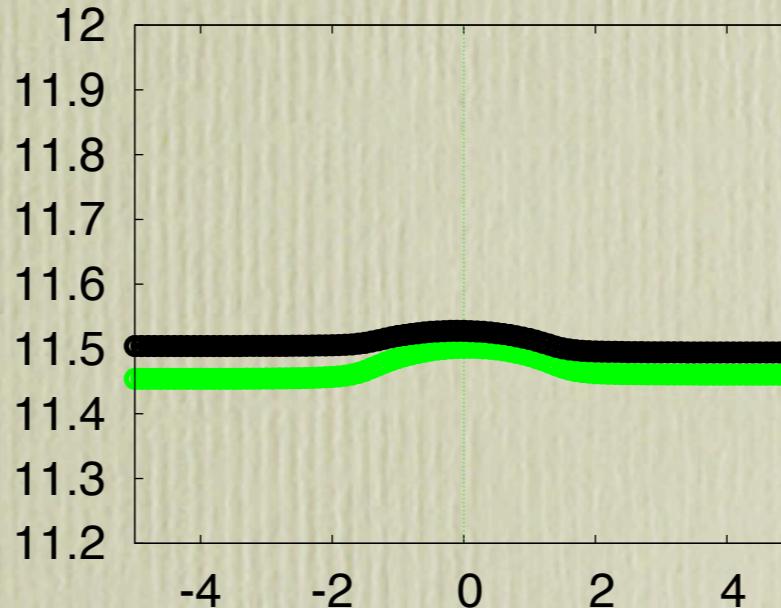
Chiral condensate in canonical ensemble

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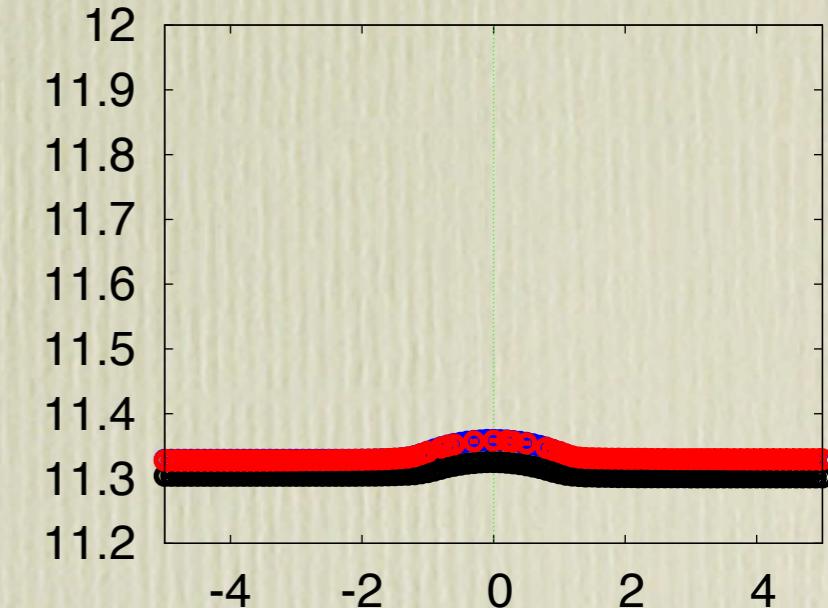
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